

Matrix representations of matroids of biased graphs correspond to gain functions

Daryl Funk*

Daniel Slilaty†

September 20, 2016

Abstract

Let M be a frame matroid or a lifted-graphic matroid and let (G, \mathcal{B}) be a biased graph representing M . Given a field \mathbb{F} , a canonical \mathbb{F} -representation of M particular to (G, \mathcal{B}) is a matrix A arising from a gain function over the multiplicative or additive group of \mathbb{F} that realizes (G, \mathcal{B}) . First, for a biased graph (G, \mathcal{B}) that is properly unbalanced, loopless, and vertically 2-connected, we show that two canonical \mathbb{F} -representations particular to (G, \mathcal{B}) are projectively equivalent iff their associated gain functions are switching equivalent. Second, when M has sufficient connectivity, we show that every \mathbb{F} -representation of M is projectively equivalent to a canonical \mathbb{F} -representation; furthermore, when (G, \mathcal{B}) is properly unbalanced, the canonical representation is particular to and unique with respect to (G, \mathcal{B}) .

1 Introduction

Frame matroids are of central importance within the class of all matroids. This was first shown by Kahn and Kung [7] who found that there are only two classes of matroid varieties that can contain 3-connected matroids: simple matroids representable over $GF(q)$ and Dowling geometries and their minors (which are frame matroids). More recently the matroid-minors project of Geelen, Gerards, and Whittle [6, Theorem 3.1] has found the following far-reaching generalization of Seymour’s decomposition theorem [9]: If \mathcal{M} is a proper minor-closed class of the class of $GF(q)$ -representable matroids, then any member of \mathcal{M} of sufficiently high vertical connectivity is either a bounded-rank perturbation of a frame matroid, the dual of a bounded-rank perturbation of a frame matroid, or is representable over some subfield of $GF(q)$. Even deeper than this, Geelen further conjectures [4] that a similar result might hold for the class of matroids not containing a $U_{a,b}$ -minor. We assume that the reader is familiar with matroid theory as in [8]. Precise definitions for technical terms used in the rest of this introduction are in Section 2.

*School of Mathematics and Statistics, Victoria University of Wellington, Wellington, New Zealand. [Email daryl.funk@vuw.ac.nz](mailto:daryl.funk@vuw.ac.nz)

†Department of Mathematics and Statistics, Wright State University, Dayton OH, USA. [Email daniel.slilaty@wright.edu](mailto:daniel.slilaty@wright.edu). Work partially supported by a grant from the Simons Foundation #246380.

Zaslavsky showed [15] that every frame matroid can be described using a graphical structure called a *biased graph*. A biased graph consists of a pair (G, \mathcal{B}) where G is a graph and \mathcal{B} is a collection of cycles of G , called *balanced*, such that no theta subgraph of G contains exactly two balanced cycles. (A *theta* graph is the union of three internally disjoint uv -paths.) Zaslavsky studies biased graphs formally in [13, 14, 16, 17]. A biased graph (G, \mathcal{B}) is said to be *balanced* when all of its cycles are balanced, *almost balanced* when there is a vertex v which all unbalanced cycles of length at least two contain, and *properly unbalanced* when it is neither balanced nor almost balanced.

Given a biased graph (G, \mathcal{B}) , its frame matroid is denoted by $F(G, \mathcal{B})$. Another matroid of importance given by (G, \mathcal{B}) is called the *lift matroid*, denoted $L(G, \mathcal{B})$. (The matroid $L(G, \mathcal{B}) = N \setminus e$ for some matroid N satisfying $N/e = M(G)$.) We also call $L(G, \mathcal{B})$ a *lifted-graphic* matroid. Zaslavsky observes that $F(G, \mathcal{B}) = L(G, \mathcal{B})$ if and only if (G, \mathcal{B}) has no two vertex-disjoint unbalanced cycles.

Given a group Γ , we will define precisely what is meant by a Γ -realization of (G, \mathcal{B}) in Section 2. For a field \mathbb{F} , let \mathbb{F}^\times and \mathbb{F}^+ be, respectively, the multiplicative and additive groups within \mathbb{F} . Given an \mathbb{F}^\times -realization φ of (G, \mathcal{B}) Zaslavsky defines [16] an \mathbb{F} -matrix $A_F(G, \varphi)$ which represents the frame matroid $F(G, \mathcal{B})$. The matrix $A_F(G, \varphi)$ is called a *frame matrix* over \mathbb{F} . Given an \mathbb{F}^+ -realization ψ of (G, \mathcal{B}) Zaslavsky again defines [16] an \mathbb{F} -matrix $A_L(G, \psi)$ which represents the lift matroid $L(G, \mathcal{B})$. The matrix $A_L(G, \psi)$ is called a *lift matrix* over \mathbb{F} . Such \mathbb{F} -representations of these two matroids are called *canonical* \mathbb{F} -representations. More specifically, we say that these canonical \mathbb{F} -representations are *particular to* or *specific to* the biased graph (G, \mathcal{B}) .

First, Conjectures 2.8 and 4.8 of Zaslavsky from [16] ask if projective equivalence of canonical representations particular to (G, \mathcal{B}) is equivalent to switching equivalence of the associated gain functions. More specifically, is it true that $A_F(G, \varphi_1)$ is projectively equivalent to $A_F(G, \varphi_2)$ iff φ_1 and φ_2 are switching equivalent? Also, is it true that $A_L(G, \psi_1)$ is projectively equivalent to $A_L(G, \psi_2)$ iff ψ_1 and ψ_2 are equivalent up to switching and scaling? We prove these statements to both be true when the biased graph (G, \mathcal{B}) is properly unbalanced, loopless, and vertically 2-connected (Theorem 5.1). Under the same conditions we also show that $A_F(G, \varphi_1)$ and $A_L(G, \psi_1)$ are never projectively equivalent. These do not hold when loosening any of these three conditions on (G, \mathcal{B}) . In this sense described we say that a canonical representation A particular to (G, \mathcal{B}) , either $A = A_F(G, \varphi)$ or $A_L(G, \psi)$, is *unique*; that is, there is no other canonical representation particular to (G, \mathcal{B}) (aside from those obtained by switching a multiplicative gain function or by switching and scaling an additive gain function) that is projectively equivalent to A .

Second, Zaslavsky conjectured [14] that every \mathbb{F} -representation A of a frame matroid or lifted graphic matroid M is projectively equivalent to a canonical representation particular to some biased graph representing M . In fact (somewhat surprisingly) the following stronger result holds.

Theorem 1.1 (Main Result I). *Let M be a 3-connected frame matroid or lifted-graphic matroid, let (G, \mathcal{B}) be a biased graph representing M , and let A be an \mathbb{F} -representation of M .*

- (1) *A is projectively equivalent to a canonical representation.*

- (2) If (G, \mathcal{B}) is properly unbalanced, then A is projectively equivalent to a canonical representation particular to (G, \mathcal{B}) .
- (3) If (G, \mathcal{B}) is properly unbalanced and loopless, then A is projectively equivalent to a canonical representation particular to and unique with respect to (G, \mathcal{B}) .

Theorem 1.1 is an immediate corollary of our other three more specific main results Theorems 5.1, 5.2, and 5.4. We note that 3-connectivity is necessary in Theorem 1.1 because frame representations and lift representations may be combined using 1- and 2-sums (see, e.g., [16, Section 5.2]) to obtain a representation that is neither of the two types.

The canonicity aspect of Theorem 1.1 but not the uniqueness aspect was proven in a simpler fashion by Geelen, Gerards, and Whittle [5] (though the result is not explicitly stated in their paper). Our proof is by induction while theirs is proven by analyzing vertex cocircuits in a cleverly defined new class of matroids called *quasi-graphic matroids*. Theorems 5.2 and 5.4 use a weaker form of connectivity that does Theorem 1.1. Other results of independent interest in this paper are Theorems 3.3, 3.4, and 3.5.

2 Preliminaries

Graphs A graph G consists of a collection of vertices $V(G)$ and a set of edges $E(G)$ where an edge has two ends each of which is attached to a vertex. A *link* is an edge that has its ends incident to distinct vertices and a *loop* is an edge that has both of its ends incident to the same vertex. The *degree* of a vertex in G is the number of ends of edges attached to that vertex and a graph is said to be *k-regular* when all of its vertices have degree k . A *path* is either a single vertex or a connected graph with two vertices of degree 1 and the remaining vertices of degree 2 each. The *length* of a path is the number of edges in it. A *cycle* is a connected 2-regular graph and the *length* of a cycle is the number of edges in it. The cycle of length n is denoted C_n . Let $\mathcal{C}(G)$ denote the set of all cycles in G . If G is a simple graph and $n \geq 2$, then by nG we mean the graph obtained from G by replacing each link by n parallel links on the same two vertices.

If $X \subseteq E(G)$, then we denote the subgraph of G consisting of the edges in X and all vertices incident to an edge in X by $G:X$. The collection of vertices in $G:X$ is denoted by $V(X)$. For $k \geq 1$, a *k-separation* of a graph is a bipartition (A, B) of the edges of G such that $|A| \geq k$, $|B| \geq k$, and $|V(A) \cap V(B)| = k$. A *vertical k-separation* (A, B) of G is a *k-separation* where $V(A) \setminus V(B) \neq \emptyset$ and $V(B) \setminus V(A) \neq \emptyset$. A graph on at least $k + 2$ vertices is said to be *vertically k-connected* when it is connected and there is no vertical r -separation for $r < k$. A graph on $k + 1$ vertices is said to be vertically k -connected when it has a spanning complete subgraph. In much of graph theory “vertically k -connected” is synonymous with just “ k -connected”. We use the modifier “vertical” to avoid confusion with other types of connectivity associated with graphs within matroid theory. A graph G that is connected and does not have a 1-separation is said to be *nonseparable*. Nonseparable graphs are always loopless and a graph with at least three vertices is nonseparable iff it is loopless and vertically 2-connected.

Given a graph G , an *oriented edge* e is an element of the edge set $E(G)$ together with a direction along it. An oriented edge e has a *head* $\mathbf{h}(e)$ and a *tail* $\mathbf{t}(e)$. As long as no

confusion may arise, we write e for both the edge $e \in E(G)$ and for e together with an implicit orientation. The reverse orientation of e is denoted e^{-1} . The collection of oriented edges of G is denoted by $\vec{E}(G)$. A *walk* w in G is a sequence of oriented edges $e_1 e_2 \cdots e_n$ for which $\mathbf{h}(e_i) = \mathbf{t}(e_{i+1})$ for each $i \in \{1, \dots, n-1\}$. The walk w is sometimes called a *uv-walk* where u is the tail of e_1 and v is the head of e_n . The *uv-walk* w is *closed* when $u = v$. The *reverse walk* of w is $w^{-1} = e_n^{-1} \cdots e_1^{-1}$.

For two graphs G and H an *isomorphism* $\iota: G \rightarrow H$ is a bijection $\iota: (V(G) \sqcup \vec{E}(G)) \rightarrow (V(H) \sqcup \vec{E}(H))$ where $\iota(V(G)) = V(H)$, $\iota(\vec{E}(G)) = \vec{E}(H)$, $\iota\mathbf{h} = \mathbf{h}\iota|_{\vec{E}(G)}$, and $\iota\mathbf{t} = \mathbf{t}\iota|_{\vec{E}(G)}$.

Given disjoint subsets $K, D \subseteq E(G)$, by $G/K \setminus D$ we mean the *minor* obtained from G by deleting the edges in D and contracting the edges in K . Given graphs G and H , we say that G has an H -minor, when there is $G/K \setminus D$ that is isomorphic to H up to deletion of isolated vertices from $G/K \setminus D$. Given a minor $G/K \setminus D$ of a graph G , one can always choose $K' \subseteq K$ such that $G:K'$ is a maximal forest of $G:K$. Hence if $D' = D \cup (K \setminus K')$, then $G/K' \setminus D' = G/K \setminus D$. We say that the minor $G/K' \setminus D'$ is obtained by contraction on an *acyclic set*.

Gain Functions Given an group Γ and a graph G , a Γ -*gain function* on G is a function $\varphi: \vec{E}(G) \rightarrow \Gamma$ satisfying $\varphi(e^{-1}) = \varphi(e)^{-1}$ when Γ is a multiplicative and $\varphi(e^{-1}) = -\varphi(e)$ when Γ is a additive. A Γ -*gain graph* is a pair (G, φ) where G is a graph and φ a Γ -gain function. Gain graphs are called “voltage graphs” within the field of topological graph theory and are sometimes called “group-labeled graphs”. A \mathbb{Z}_2 -gain graph is a *signed graph*. Given any walk $e_1 \cdots e_n$ we define $\varphi(e_1 \cdots e_n) = \varphi(e_1) \cdots \varphi(e_n)$ for multiplicative groups and $\varphi(e_1 \cdots e_n) = \varphi(e_1) + \cdots + \varphi(e_n)$ for additive groups. These yield $\varphi(w^{-1}) = \varphi(w)^{-1}$ and $\varphi(w^{-1}) = -\varphi(w)$, respectively, for any walk w and also $\varphi(w_1 w_2) = \varphi(w_1) \varphi(w_2)$ and $\varphi(w_1 w_2) = \varphi(w_1) + \varphi(w_2)$, respectively, for any *uv-walk* w_1 and *vz-walk* w_2 .

If C is a cycle in G , then let w_C be a closed Eulerian walk along C . Of course, w_C is only well defined up to a choice of starting vertex and direction around C . Now define a cycle C in G to be *balanced* with respect to φ when $\varphi(w_C)$ is the identity and let \mathcal{B}_φ be the collection of cycles in G that are balanced with respect to φ .

Given a Γ -gain function φ on G and a function $\eta: V(G) \rightarrow \Gamma$, define the gain function φ^η by $\varphi^\eta(e) = \eta(\mathbf{t}(e))^{-1} \varphi(e) \eta(\mathbf{h}(e))$ for multiplicative groups and $\varphi^\eta(e) = -\eta(\mathbf{t}(e)) + \varphi(e) + \eta(\mathbf{h}(e))$ for additive groups. We call η a *switching function*. Note that a cycle C is balanced with respect to φ iff C is balanced with respect to φ^η , i.e., $\mathcal{B}_\varphi = \mathcal{B}_{\varphi^\eta}$. Note that for switching functions η_1 and η_2 that $(\varphi^{\eta_1})^{\eta_2} = \varphi^{\eta_1 \eta_2}$ or $\varphi^{\eta_1 + \eta_2}$.

When two Γ -gain functions φ and ψ satisfy $\varphi^\eta = \psi$ for some η , we say that φ and ψ are *switching equivalent*. For a field \mathbb{F} , when the group $\Gamma = \mathbb{F}^+$, we say that φ and ψ are equivalent up to *switching and scaling* when there is a switching function η and scalar $a \in \mathbb{F}^\times$ such that $a\varphi^\eta = \psi$. Propositions 2.1 and 2.2 are immediate.

Proposition 2.1. *Let F be a maximal forest of a graph G and φ a Γ -gain function on G . There is switching function η such that $\varphi^\eta(e)$ is the identity for all oriented edges e in F .*

Given a maximal forest F of G , a Γ -gain function φ is said to be *F-normalized* when $\varphi(e)$ is the identity for all oriented edges e in F .

Proposition 2.2. *Let G be a graph, F a maximal forest of G , and φ and ψ two F -normalized Γ -gain functions on G . Then $\varphi = \psi^\eta$ for some η iff $\varphi = \psi$.*

Given a Γ -gain function φ on a graph G and a minor $G' = G/K \setminus D$ of G , we wish to give an *induced* Γ -gain function $\varphi|_{G'}$. If e is an edge of G and $G' = G \setminus e$, then $\varphi|_{G'}$ is defined on $G \setminus e$ by restriction. If e is a link and $G' = G/e$, then $\varphi|_{G'}$ is defined up to switching as follows. Since e is a link and not a loop, there is switching function η , such that $\varphi^\eta(e) = 1$. Now $\varphi|_{G'}$ is defined up to switching by restriction of φ^η to $E(G) \setminus e$. With ordinary graphs, if e is a loop, then $G/e = G \setminus e$. In the context of biased graphs there are other considerations with contractions of unbalanced loops. We will address the contraction of unbalanced loops in the next part of this section. However, for $G' = G/K \setminus D$, we can define $\varphi|_{G'}$ (up to switching) iteratively when we only delete loops rather than contract them. One can define $\varphi|_{G'}$ globally (again, up to switching) as follows. Let $G:K'$ be a maximal forest in $G:K$ and let $D' = D \cup (K \setminus K')$ and we have $G/K \setminus D = G/K' \setminus D'$. Let F be a maximal forest of G whose edges contain K' . Let φ^η be the F -normalization of φ and so we define $\varphi|_{G'}$ by restricting φ to $E(G') = E(G) \setminus (K \cup D)$.

Proposition 2.3. *Let G be a graph, let F be the edge set of a forest in G , and let Γ be an abelian group. If φ and ψ are switching inequivalent Γ -gain functions on G , then $\varphi|_{G/F}$ and $\psi|_{G/F}$ are switching inequivalent.*

Proof. Extend F to a maximal forest F_m in G and assume that φ and ψ are normalized on F_m . Since φ and ψ are switching inequivalent, certainly $\varphi \neq \psi$, and since both are normalized on F_m , their restrictions to $E(G) \setminus F_m$ are not equal. Now in G/F , the induced gain functions $\varphi|_{G/F}$ and $\psi|_{G/F}$ are normalized on maximal forest $F_m \setminus F$ of G/F . Furthermore, $\varphi|_{G/F} \neq \psi|_{G/F}$ which implies that $\varphi|_{G/F}$ and $\psi|_{G/F}$ are switching inequivalent on G/F by Proposition 2.2. \square

Biased Graphs A *biased graph* is a pair (G, \mathcal{B}) where G is a graph and \mathcal{B} is a collection of cycles in G (called *balanced*) for which any theta subgraph contains either 0, 1, or 3 cycles from \mathcal{B} . That is, no theta subgraph contains exactly two cycles from \mathcal{B} . We say such a collection \mathcal{B} satisfies the *theta property*. In the language of matroids, \mathcal{B} is *linear class* of circuits of the cycle matroid $M(G)$. A biased graph (G, \mathcal{B}) is *balanced* when \mathcal{B} contains all cycles of G , and is otherwise *unbalanced*; (G, \mathcal{B}) is *contrabalanced* if \mathcal{B} is empty. An unbalanced loop is called a *joint*. A set of edges X in (G, \mathcal{B}) (or a subgraph H of G) is said to be *balanced* when every cycle in $G:X$ (or in H) is balanced. A vertex v in a biased graph (G, \mathcal{B}) is called a *balancing vertex* when every unbalanced cycle contains v . A biased graph is *almost balanced* if it has a balancing vertex after deleting its loops. An unbalanced biased graph that is not almost balanced is said to be *properly unbalanced*. A *simple biased graph* is a biased graph without balanced cycles of length 1 or 2 and without two joints at the same vertex. A simple biased graph need not have an underlying graph that is simple. We often write $\Omega = (G, \mathcal{B})$ and speak of the biased graph Ω when there is no need to be explicit about the underlying graph G and its collection of balanced cycles \mathcal{B} .

For two biased graphs (G, \mathcal{B}) and (H, \mathcal{S}) an *isomorphism* $\iota: (G, \mathcal{B}) \rightarrow (H, \mathcal{S})$ consists of an underlying graph isomorphism $\iota: G \rightarrow H$ that takes \mathcal{B} to \mathcal{S} . A biased graph (G, \mathcal{B}) is

said to be vertically k -connected when its underlying graph is vertically k -connected. The prime example of a biased graph is given in Proposition 2.4.

Proposition 2.4 (Zaslavsky [13]). *If φ is a Γ -gain function on a graph G , then (G, \mathcal{B}_φ) is a biased graph.*

Given a biased graph (G, \mathcal{B}) and a group Γ , a Γ -realization of (G, \mathcal{B}) is a Γ -gain function φ for which $\mathcal{B}_\varphi = \mathcal{B}$.

Let (G, \mathcal{B}) be a biased graph and e an edge in G . Define $(G, \mathcal{B}) \setminus e = (G \setminus e, \mathcal{B}|_{G \setminus e})$ where $\mathcal{B}|_{G \setminus e} = \mathcal{B} \cap \mathcal{C}(G \setminus e)$. If e is a link, then define $(G, \mathcal{B})/e = (G/e, \mathcal{B}|_{G/e})$ where $\mathcal{B}|_{G/e} = \{C \in \mathcal{C}(G/e) : C \in \mathcal{B} \text{ or } C \cup e \in \mathcal{B}\}$. If e is a balanced loop, then $(G, \mathcal{B})/e = (G, \mathcal{B}) \setminus e$. When e is an unbalanced loop on vertex v , then $(G, \mathcal{B})/e = (G', \mathcal{B}')$ where G' is obtained from G by taking each loop $e' \neq e$ incident to v and making it balanced (if it isn't already) and taking each link f incident to v and replacing it with an unbalanced loop attached to its other endpoint. The set \mathcal{B}' is \mathcal{B} restricted to the subgraph $G - v$ along with any new balanced loops incident to v . A *minor* of (G, \mathcal{B}) is a biased graph obtained from (G, \mathcal{B}) by deletions and contractions of edges and deletions of isolated vertices. A *link minor* of (G, \mathcal{B}) is a minor that is obtained without contracting any unbalanced loops. Thus a link minor $(G, \mathcal{B})/K \setminus D$ must always satisfy that K is a balanced edge set and so $(G, \mathcal{B})/K \setminus D = (G', \mathcal{B}|_{G'})$ for $G' = G/K \setminus D$. Also we get that $(G, \mathcal{B})/K \setminus D = (G, \mathcal{B})/K' \setminus D'$ for some $G:K'$ that is a maximal forest in $G:K$ and $D' = D \cup (K \setminus K')$, that is, the link minor $(G', \mathcal{B}|_{G'}) = (G, \mathcal{B})/K' \setminus D'$ can always be obtained by contraction with an acyclic set. We say that biased graph (G, \mathcal{B}) has an (H, \mathcal{S}) -link minor (respectively an (H, \mathcal{S}) -minor) when there is link minor (respectively minor) $(G', \mathcal{B}|_{G'})$ of (G, \mathcal{B}) that is isomorphic to (H, \mathcal{S}) up to deletion of isolated vertices in $(G', \mathcal{B}|_{G'})$.

Given a Γ -realization φ of biased graph (G, \mathcal{B}) and a link minor $(G', \mathcal{B}|_{G'})$ of (G, \mathcal{B}) , we immediately get Proposition 2.5 for link minors but not quite yet for general minors. For an unbalanced loop e with endpoint v , since $(G, \mathcal{B})/e$ is just a the subgraph $G - v$ along with some loops added, we can define $\varphi|_{G'}$ using the same notion for subgraphs with the addition that any new unbalanced loops have a non-identity gain value arbitrarily assigned to them and balanced loops have the identity gain assigned to them. We now get Proposition 2.5. We call the Γ -realization $\varphi|_{G'}$ of $(G', \mathcal{B}|_{G'})$ the *induced* Γ -realization of $(G', \mathcal{B}|_{G'})$.

Proposition 2.5. *If φ is a Γ -realization of (G, \mathcal{B}) and $(G', \mathcal{B}|_{G'})$ is a minor of (G, \mathcal{B}) , then the induced gain function $\varphi|_{G'}$ is a Γ -realization of $(G', \mathcal{B}|_{G'})$.*

Proposition 2.6. *If φ is a Γ -realization of (G, \mathcal{B}) and $\iota: (H, \mathcal{S}) \rightarrow (G, \mathcal{B})$ is an isomorphism, then $\varphi \iota$ is Γ -realization of (H, \mathcal{S}) .*

Matroids of biased graphs There are two matroids normally associated with a biased graph (G, \mathcal{B}) : the *frame matroid* $F(G, \mathcal{B})$ and the *lift matroid* $L(G, \mathcal{B})$. The lift matroid $L(G, \mathcal{B})$ also has the closely related *complete lift matroid* $L_0(G, \mathcal{B})$ where $L_0(G, \mathcal{B}) \setminus e_0 = L(G, \mathcal{B})$. A full introduction to the basic properties of these matroids can be found in [14].

The set of elements of the frame matroid $F(G, \mathcal{B})$ is the set of edges $E(G)$. The rank function for $F(G, \mathcal{B})$ is defined for $X \subseteq E(G)$ by $r(X) = |V(X)| - b_X$ in which b_X is the number of balanced components in $G:X$. A subset $C \subseteq E(G)$ is a circuit of $F(G, \mathcal{B})$ when

$G:C$ is a balanced cycle or $G:C$ is a subdivision of one of the subgraphs shown in Figure 1 and contains no balanced cycles. For any edge e , we get that $F(G, \mathcal{B}) \setminus e = F((G, \mathcal{B}) \setminus e)$ and $F(G, \mathcal{B})/e = F((G, \mathcal{B})/e)$.

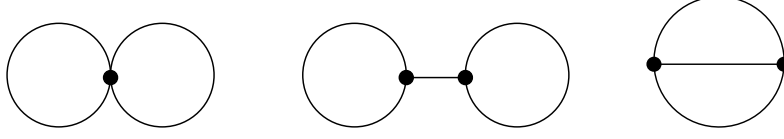


Figure 1: Circuits of the frame matroid.

The set of elements of the lift matroid $L(G, \mathcal{B})$ is the set of edges $E(G)$. The rank function for $L(G, \mathcal{B})$ is defined for $X \subseteq E(G)$ by $r(X) = |V(X)| - c_X + \epsilon_X$ in which c_X is the number of components in $G:X$ and $\epsilon_X = 1$ when X is unbalanced where $\epsilon_X = 0$ when X is balanced. A subset $C \subseteq E(G)$ is a circuit of $L(G, \mathcal{B})$ when $G:C$ is a balanced cycle or $G:C$ is a subdivision of one of the subgraphs shown in Figure 2 and contains no balanced cycles. Let G_0 be the graph obtained from G by adding a loop, call it e_0 , to any vertex or to a new vertex. The complete lift matroid $L_0(G, \mathcal{B})$ is defined as $L(G_0, \mathcal{B})$. Note that $L_0(G, \mathcal{B})/e_0 = M(G)$, the ordinary graphic matroid. For any edge e , we get that $L_0(G, \mathcal{B}) \setminus e = L_0((G, \mathcal{B}) \setminus e)$ and for any link e , $L_0(G, \mathcal{B})/e = L_0((G, \mathcal{B})/e)$.

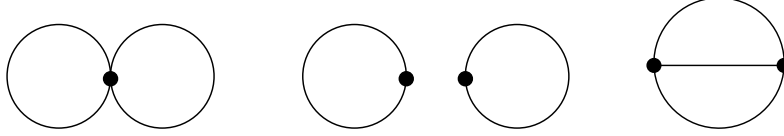


Figure 2: Circuits of the lift matroid.

Canonical frame representations Given the field \mathbb{F} , let \mathbb{F}^\times denote the multiplicative group of \mathbb{F} .

Let G be a graph and for each vertex $v \in V(G)$ let \widehat{v} denote the column vector with rows indexed by $V(G)$ having a 1 in row corresponding to v and a zero in every other row. Now given a gain function $\varphi: G \rightarrow \mathbb{F}^\times$, define the *frame matrix* $A_F(G, \varphi)$ as follows. The rows of $A_F(G, \varphi)$ are indexed by $V(G)$ and columns by $E(G)$. Arbitrarily choose some orientation for each link $e \in E(G)$. Now the column of $A_F(G, \varphi)$ corresponding to $e \in E(G)$ is defined as

- $\widehat{\mathbf{t}(e)} - \varphi(e)\widehat{\mathbf{h}(e)}$ when e is a link, and
- $\widehat{\mathbf{h}(e)}$ when e is a joint, and
- $\mathbf{0}$ when e is a balanced loop.

When e is a link, note that using e^{-1} rather than e to define the column yields

$$\widehat{\mathbf{t}(e^{-1})} - \varphi(e^{-1})\widehat{\mathbf{h}(e^{-1})} = \widehat{\mathbf{h}(e)} - \varphi(e)^{-1}\widehat{\mathbf{t}(e)} = -\varphi(e)^{-1}(\widehat{\mathbf{t}(e)} - \varphi(e)\widehat{\mathbf{h}(e)}).$$

Thus the arbitrary choice of orientations for the links only changes $A_F(G, \varphi)$ up to column scaling which, of course, does not affect that matroid defined by the matrix $A_F(G, \varphi)$.

Theorem 2.7 (Zaslavsky [16]). *If G is a graph and φ a \mathbb{F}^\times -gain function on G , then the vector matroid of the frame matrix $A_F(G, \varphi)$ is equal to the frame matroid $F(G, \mathcal{B}_\varphi)$. Furthermore, if η is a switching function, then the matrices $A_F(G, \varphi)$ and $A_F(G, \varphi^\eta)$ are projectively equivalent.*

Canonical lift representations Given the field \mathbb{F} , let \mathbb{F}^+ denote the additive subgroup of \mathbb{F} . Now given a graph G and a gain function $\psi: G \rightarrow \mathbb{F}^+$, define the *complete lift matrix* $A_{L_0}(G, \psi)$ as follows. The rows of $A_F(G, \psi)$ are indexed by $V(G) \cup v_0$ and columns by $E(G) \cup e_0$. Arbitrarily choose some orientation for each link $e \in E(G)$. Now the column of $A_{L_0}(G, \psi)$ corresponding to $e \in E(G) \cup e_0$ is defined as

- $\widehat{\mathbf{t}(e)} - \widehat{\mathbf{h}(e)} + \psi(e)\widehat{v}_0$ when e is a link, and
- \widehat{v}_0 when e is a joint or $e = e_0$, and
- $\mathbf{0}$ when e is a balanced loop.

When e is a link, note that using e^{-1} rather than e to define the column yields

$$\widehat{\mathbf{t}(e^{-1})} - \widehat{\mathbf{h}(e^{-1})} + \psi(e^{-1})\widehat{v}_0 = -(\widehat{\mathbf{t}(e)} - \widehat{\mathbf{h}(e)} + \psi(e)\widehat{v}_0)$$

Thus the arbitrary choice of orientations for the links only changes $A_{L_0}(G, \psi)$ up to column scaling which, of course, does not affect that matroid defined by the matrix $A_{L_0}(G, \psi)$. Define the lift matrix $A_L(G, \psi)$ as the matrix obtained from the complete lift matrix $A_{L_0}(G, \psi)$ by deleting the column corresponding to e_0 .

Theorem 2.8 (Zaslavsky [16]). *If G is a graph and ψ a \mathbb{F}^+ -gain function on G , then the vector matroid of the complete lift matrix $A_{L_0}(G, \psi)$ is equal to the complete lift matroid $L_0(G, \mathcal{B}_\psi)$. Furthermore, if η is a switching function and $a \in \mathbb{F}^\times$, then matrices $A_F(G, \psi)$ and $A_F(G, a\psi^\eta)$ are projectively equivalent.*

Subdivisions A *subdivision* of a biased graph (G, \mathcal{B}) is a biased graph (H, \mathcal{S}) in which H is a subdivision of G where a cycle C of H is in \mathcal{S} if and only if its corresponding cycle C' of G is in \mathcal{B} .

Proposition 2.9. *Let (H, \mathcal{S}) be a subdivision of (G, \mathcal{B}) .*

1. *The \mathbb{F} -representations of $F(H, \mathcal{S})$ are in one-to-one correspondence with the \mathbb{F} -representations of $F(G, \mathcal{B})$ up to projective equivalence.*
2. *The \mathbb{F} -representations of $L(H, \mathcal{S})$ are in one-to-one correspondence with the \mathbb{F} -representations of $L(G, \mathcal{B})$ up to projective equivalence.*
3. *The Γ -realizations of (H, \mathcal{S}) are in one-to-one correspondence with the Γ -realizations of (G, \mathcal{B}) up to switching.*

Proof. The first two parts are because two edges incident to a vertex of degree two form a coparallel pair of elements of in $F(H, \mathcal{S})$ and in $L(H, \mathcal{S})$. The third part is evident. \square

Rolling and unrolling for almost-balanced biased graphs Let (G, \mathcal{B}) be a biased graph with balancing vertex u and $\delta(u) = \{e_1, \dots, e_k\}$. By the theta property, for each pair e_i, e_j either all cycles containing both e_i and e_j are balanced or all cycles containing both e_i and e_j are unbalanced. This yields an equivalence relation \sim on $\delta(u)$ in which $e_i \sim e_j$ if there is a balanced cycle containing $\{e_i, e_j\}$. Let J be the set of joints of an almost-balanced biased graph (G, \mathcal{B}) with balancing vertex u of $(G, \mathcal{B}) \setminus J$. Let $J' \subseteq J$ be the set of joints not incident to u and J'' be the set of joints incident to u . (Normally one can assume that $|J''| \leq 1$.) Each pair of edges in $\Sigma = \delta(u) \cup J'$ behaves in $F(G, \mathcal{B})$ in an analogous manner as pairs of edges in $\delta(u)$, that is, for each pair of edges $f_1, f_2 \in \Sigma$, either every minimal path linking the endpoints of f_1 and f_2 in $G - u$ together with $\{f_1, f_2\}$ forms a circuit in $F(G, \mathcal{B})$ or all such paths together with $\{f_1, f_2\}$ are independent sets in $F(G, \mathcal{B})$. Indeed, this defines an equivalence relation on Σ consistent with that previously defined on $\delta(u)$. We call these equivalence classes of Σ its *unbalancing classes*. Now consider the biased graph (G', \mathcal{B}') obtained from (G, \mathcal{B}) as follows: replace each joint $e \in J'$ incident to a vertex $v \neq u$ with a uv -link and define \mathcal{B}' to be those cycles having intersection of size 0 or 2 with each unbalancing class of Σ . Funk has noted [3, Proposition 1.25] that $F(G', \mathcal{B}') = F(G, \mathcal{B})$. We say that (G, \mathcal{B}) is a *roll-up* of (G', \mathcal{B}') at vertex u and (G', \mathcal{B}') is the *unrolling* of (G, \mathcal{B}) at vertex u . It is also worth noting that $F(G, \mathcal{B}) = F(G', \mathcal{B}') = L(G', \mathcal{B}')$ whereas $L(G, \mathcal{B}) \neq L(G', \mathcal{B}')$.

If a biased graph has two distinct balancing vertices, then it has a very restricted form as given in Proposition 2.10. It is worth noting that if any G_i in Proposition 2.10 has more than one edge, then the lift and frame matroids of Ω are not 3-connected. Note that, given one of the balancing vertices v of Ω , the edges of the subgraphs $G_1 \cup \dots \cup G_m$ incident to v form the unbalancing classes at v .

Proposition 2.10 (Zaslavsky [12]). *Let Ω be a vertically 2-connected, unbalanced biased graph with two distinct vertices x and y such that each is a balancing vertex. Then Ω is a union of biased subgraphs $G_1 \cup \dots \cup G_m$ where for each pair $i \neq j$, $G_i \cap G_j = \{x, y\}$, and a cycle of Ω is balanced if and only if it is contained in a single subgraph G_i . Furthermore, if $m \geq 3$ then x and y are the only balancing vertices of Ω .*

Given a biased graph Ω of the form given in Proposition 2.10 with $m \geq 3$, a *double roll-up* Ω' of Ω is obtained as follows. Given $i \neq j$, let E_i be the edges of G_i incident to balancing vertex x and let E_j be the edges of G_j incident to balancing vertex y . Replace each edge of E_i with an unbalanced loop incident to its other endpoint and replace each edge of E_j with an unbalanced loop incident to its other endpoint. One can check that $F(\Omega) = F(\Omega')$. The reverse operation is called a *double unrolling*.

ΔY - and $Y\Delta$ -exchanges. Consider a labeled K_4 containing edges $X = \{a, b, c\}$ forming a triangle and edges $Y = \{a', b', c'\}$ forming a $K_{1,3}$ -subgraph for which $\{a, a'\}$, $\{b, b'\}$, $\{c, c'\}$ are all 2-edge matchings. Given a graph G containing a triangle with edges X , a ΔY -exchange replaces edges X in G with the edges Y incident to a new vertex. We then drop the “prime” marks on the replaced edges by identifying a with a' , b with b' , and c with c' . This new graph is denoted $\Delta_X G$.

The inverse operation, a $Y\Delta$ -exchange, is performed as follows. Given a graph G containing a $K_{1,3}$ -subgraph on edges $\{a, b, c\}$, relabel the edges as $Y = \{a', b', c'\}$. The $Y\Delta$ -exchange

on G replaces the edges Y in G with those of the triangle $X = \{a, b, c\}$ with edges labeled and identified as above, and deletes the degree-3 vertex of the $K_{1,3}$ -subgraph. This new graph is denoted $\nabla_Y G$.

Let (G, \mathcal{B}) be a biased graph which has a balanced 3-cycle X . Define the ΔY -exchange $\Delta_X(G, \mathcal{B}) = (\Delta_X G, \Delta_X \mathcal{B})$ where $\Delta_X \mathcal{B}$ is the set

$$\{C \in \mathcal{B} : |C \cap X| = 0 \text{ or } 2\} \cup \{C \Delta X : C \in \mathcal{B} \text{ and } |C \cap X| = 1\}$$

Proposition 2.11. $\Delta_X(G, \mathcal{B}) = (\Delta_X G, \Delta_X \mathcal{B})$ is a biased graph.

Proof. Given a theta subgraph Θ of $\Delta_X(G, \mathcal{B})$, $|\Theta \cap X| \in \{0, 2, 3\}$. If $|\Theta \cap X| = 0$, then Θ is also a theta subgraph of (G, \mathcal{B}) and so does not have exactly two cycles in $\Delta_X \mathcal{B}$. If $|\Theta \cap X| \neq 0$, then the cycles of Θ correspond via the ΔY -operation to the cycles of a theta subgraph Θ_2 of G . Since triangle X in (G, \mathcal{B}) is balanced, any cycle C of (G, \mathcal{B}) containing an edge e of X has the same bias as the cycle $C \Delta X$ of $\Delta_X(G, \mathcal{B})$. Thus there is not exactly two cycles of Θ in $\Delta_X \mathcal{B}$. \square

Similarly, a $Y\Delta$ exchange on a $K_{1,3}$ -subgraph with edges Y in (G, \mathcal{B}) , denoted by $\nabla_Y(G, \mathcal{B}) = (\nabla_Y G, \nabla_Y \mathcal{B})$ in which $\nabla_Y \mathcal{B}$ is defined as

$$\{Y\} \cup \min\{C, C \Delta Y : C \in \mathcal{B} \text{ and } |C \cap Y| = 0 \text{ or } 2\}.$$

Proposition 2.12. $\nabla_Y(G, \mathcal{B}) = (\nabla_Y G, \nabla_Y \mathcal{B})$ is a biased graph

Proof. Let Θ be a theta subgraph of $\nabla_Y(G, \mathcal{B})$. If Θ contains the triangle Y , then by the definition of the $Y\Delta$ -operation, the other two cycles of Θ have the same bias and so we do not have exactly two balanced cycles in Θ . If Θ contains one or two edges from Y , then Θ corresponds to a theta subgraph Θ_2 of G . Thus there are not exactly two cycles of Θ in $\nabla_Y \mathcal{B}$. \square

A consequence of our definitions is Proposition 2.13, which can be proven by comparing flats of two matroids.

Proposition 2.13. Let (G, \mathcal{B}) be a biased graph.

1. Let X be a balanced 3-cycle in (G, \mathcal{B}) . Then

$$F(\Delta_X(G, \mathcal{B})) = \Delta_X F(G, \mathcal{B}) \text{ and } L_0(\Delta_X(G, \mathcal{B})) = \Delta_X L_0(G, \mathcal{B}).$$

2. Let Y be a $K_{1,3}$ -subgraph in (G, \mathcal{B}) . Then

$$F(\nabla_X(G, \mathcal{B})) = \nabla_X F(G, \mathcal{B}) \text{ and } L_0(\nabla_X(G, \mathcal{B})) = \nabla_X L_0(G, \mathcal{B}).$$

We also need to carefully consider matrix representations and $Y\Delta$ - and ΔY -exchanges. Over any field \mathbb{F} , an \mathbb{F} -matrix representing the matroid $M(K_4)$ is projectively equivalent to the matrix $I(K_4)$ shown below, perhaps with one row omitted, and/or with zero rows added.

$$I(K_4) = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & -1 \\ -1 & -1 & -1 & 0 & 0 & 0 \end{bmatrix}$$

The first three columns of this matrix represent a $K_{1,3}$ -subgraph for K_4 and the last three columns a triangle for K_4 .

Given an \mathbb{F} -matrix A' such that $M(A')$ contains a triangle X , A' is projectively equivalent to an \mathbb{F} -matrix A in which the columns corresponding to X are as the last three columns of $I(K_4)$, perhaps with one row omitted, and/or with zero rows added. The matrix $\Delta_X A$ is obtained from the matrix A by replacing the columns for X with first three columns of $I(K_4)$, save perhaps the final row of the first three columns if full rank is necessary. From [1] $M(\Delta_X A) = \Delta_X M(A)$. Similarly if $M(A')$ contains a triad X , then A' is projectively equivalent to an \mathbb{F} -matrix A in which the columns corresponding to X are as the first three columns of $I(K_4)$ (again with any necessary adjustments for three rows or more than three rows in these columns). The matrix $\nabla_X A$ is obtained from the matrix A by replacing the columns of X with those of the last three columns of $I(K_4)$. From [1] $M(\nabla_X A) = \nabla_X M(A)$. Proposition 2.14 follows immediately when switching the gain function so that the edges of the triad or balanced triangle all have the identity gain.

Proposition 2.14. *Let A be an \mathbb{F} -representation of matroid M ,*

1. *If $A_F(G, \varphi)$ is a frame matrix and X is a $K_{1,3}$ subgraph of G , then $\nabla_X A_F(G, \varphi)$ is a frame matrix with underlying graph $\nabla_X G$*
2. *If $A_F(G, \varphi)$ is a frame matrix and X is a balanced triangle in (G, \mathcal{B}_φ) , then $\Delta_X A_F(G, \varphi)$ is a frame matrix with underlying graph $\Delta_X G$.*
3. *If $A_L(G, \psi)$ is a lift matrix and X is a $K_{1,3}$ subgraph of G , then $\nabla_X A_L(G, \psi)$ is a lift matrix with underlying graph $\nabla_X G$.*
4. *If $A_L(G, \psi)$ is a lift matrix and X is a balanced triangle in (G, \mathcal{B}_ψ) , then $\Delta_X A_L(G, \psi)$ is a lift matrix with underlying graph $\Delta_X G$.*

Matrix representations of matroids behave well with respect to ΔY - and $Y\Delta$ -exchanges.

Proposition 2.15 (Whittle [11, Lemma 5.7]). *Let X be a triangle of matroid M . The projective equivalence classes of \mathbb{F} -representations of M are in one-to-one correspondence with the projective equivalence classes of \mathbb{F} -representations of $\Delta_X M$.*

Let (G, \mathcal{B}) be a biased graph, and let φ is a Γ -gain function on G realizing (G, \mathcal{B}) . Let X be a balanced triangle or $K_{1,3}$ subgraph of (G, \mathcal{B}) . By switching we may assume φ is the identity on X , so φ realizes $(\Delta_X G, \Delta_X \mathcal{B})$ or $(\nabla_X G, \nabla_X \mathcal{B})$. Proposition 2.16 can be thought of as an analogue to Proposition 2.15.

Proposition 2.16. *If (G, \mathcal{B}) is a biased graph containing a balanced triangle T and Γ is an abelian group, then up to switching the Γ -realizations of (G, \mathcal{B}) are in one-to-one correspondence with the Γ -realizations of $\Delta_X(G, \mathcal{B})$.*

Proof. There exists a maximal forest F of $\Delta_X G$ that contains all three edges of X . Now for any $e \in X$, $F - e$ is a maximal forest of G that contains two edges of X . Any Γ -realization of (G, \mathcal{B}) is switching equivalent to a unique $(F - e)$ -normalized Γ -realization and any Γ -realization of $\Delta_X(G, \mathcal{B})$ is switching equivalent to a unique F -normalized Γ -realization. Notice also that any $(F - e)$ -normalized Γ -realization of (G, \mathcal{B}) also has identity gain value on e . Now, evidently, φ is a Γ -realization of (G, \mathcal{B}) iff the same function φ is also a Γ -realization of $\Delta_X(G, \mathcal{B})$. \square

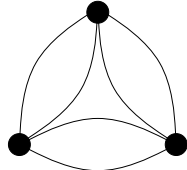
3 The minor-minimal, vertically 2-connected, properly unbalanced biased graphs

Let \mathcal{G}_0 denote the set of link-minor-minimal biased graphs that are vertically 2-connected and properly unbalanced. We first describe 13 biased graphs in \mathcal{G}_0 and then show that these 13 biased graphs form the complete set. The graph $2C_4''$ (which we call the *tube* graph) is obtained from an unlabeled 4-cycle by doubling each edge in a pair of opposite edges. Three of the biased graphs in \mathcal{G}_0 have underlying graph $2C_4''$, four have underlying graph K_4 , and six have underlying graph $2C_3$. We call the biased graphs in \mathcal{G}_0 our *base biased graphs*.

Biased K_4 's. The set of cycles of the graph K_4 consists of four triangles and three quadrilaterals. There are seven isomorphism classes of biased graphs of the form (K_4, \mathcal{B}) (see [13]), which are jovially named the *Seven Dwarves*. We denote by $D_{t,q} = (K_4, \mathcal{B}_{t,q})$ the biased K_4 with exactly t balanced triangles and q balanced quadrilaterals. The seven biased K_4 's are $D_{0,0}$, $D_{0,1}$, $D_{0,2}$, $D_{0,3}$, $D_{1,0}$, $D_{2,1}$, and $D_{4,2}$. A biased K_4 is properly unbalanced if and only if it does not contain a balanced triangle. These are $D_{0,0}$, $D_{0,1}$, $D_{0,2}$, and $D_{0,3}$.

Biased $2C_3$'s. An biased graph of the form $(2C_3, \mathcal{B})$ is properly unbalanced if and only if it does not contain a balanced 2-cycle.

Proposition 3.1. *There are six unlabeled biased $2C_3$'s without a balancing vertex. These are shown in Figure 3.*








Name	Balanced cycles
T_0	none
T_1	
T_2	
T'_2	
T_3	
T_4	

Figure 3: The graph $2C_3$ and its six possible classes of balanced cycles not containing a cycle of length two.

Proof of Proposition 3.1. The collection of balanced triangles in a biased graph $(2C_3, \mathcal{B})$ without a balanced 2-cycle is any collection of triangles for which no two triangles intersect in more than a single edge. There are eight triangles in $2C_3$ and any five must contain a pair that intersect in more than one edge. Now for $k \in \{0, 1, 2, 3, 4\}$, the reader can check that the only possible configurations of k balanced triangles are those in the table. \square

Note that $\Delta_X T_i \cong D_{0,i-1}$ for $i \in \{1, 2, 3, 4\}$ and $\Delta_X T'_2 \cong D_{1,0}$.

Biased $2C_4'''$'s. A biased tube is properly unbalanced if and only if it has no balanced 2-cycle. There are three such tubes, described in Figure 4. They are all in \mathcal{G}_0 .

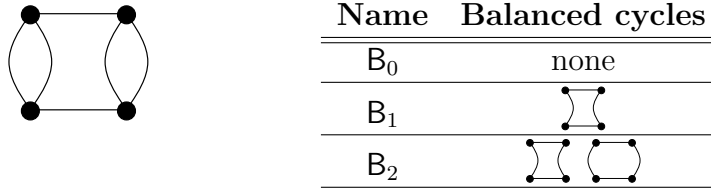


Figure 4: The graph $2C_4'''$ and its three possible classes of balanced cycles not containing a cycle of length two.

Proposition 3.2 follows immediately from Menger's Theorem.

Proposition 3.2. *If (G, \mathcal{B}) is vertically 2-connected and contains two vertex disjoint unbalanced cycles neither of which is a loop, then (G, \mathcal{B}) contains a subdivision of B_0 , B_1 , or B_2 .*

It remains just to show that a properly unbalanced biased graph without two vertex-disjoint unbalanced cycles has a link minor from \mathcal{G}_0 . Such biased graphs are called *tangled*. The structure of tangled signed graphs was characterized by Slilaty [10] and the structure of tangled biased graphs in general was characterized by Chen and Pivotto [2]. Theorem 3.3 could be proven (although not trivially) as a consequence of Chen and Pivotto's work in [2] but the direct proof here seems no more difficult.

Theorem 3.3. *If Ω is a tangled biased graph, then Ω contains a link minor that is a biased K_4 with no balanced triangle or a biased $2C_3$ with no balanced 2-cycle.*

Proof. Let Ω be a tangled biased graph that is a link-minor minimal counterexample to our result. If Ω has more than one unbalanced block but no two disjoint unbalanced cycles, then Ω must have a balancing vertex, a contradiction. Hence Ω has only one unbalanced block. Evidently our desired minor exists in Ω iff it exists in the unbalanced block of Ω and so the minimality of Ω therefore implies that it is nonseparable. By minimality we may also assume that Ω has no balanced 2-cycles. Now Ω cannot be tangled unless Ω has at least three vertices and so we assume that Ω is vertically 2-connected, loopless, has at least three vertices, and no balanced 2-cycles. By Claim 1, we now get that the underlying graph of Ω is simple.

Claim 1. The underlying graph of Ω is simple.

Proof of Claim: By way of contradiction assume that C is an unbalanced 2-cycle in Ω with vertices x and y . Thus $\Omega - x$ contains an unbalanced cycle C_y passing through y and $\Omega - y$ contains an unbalanced cycle C_x passing through x . Since Ω is tangled, $C_x \cup C_y \cup C$ contains a biased $2C_3$ without a balanced 2-cycle, a contradiction. ♣

If Ω has just three vertices, then the underlying graph of Ω is a triangle; however, now Ω has a balancing vertex, a contradiction. Now suppose Ω has exactly four vertices. For any vertex v , $\Omega - v$ must be an unbalanced triangle and so Ω is a biased K_4 without a balanced

triangle, a contradiction. So for the remainder of the proof we can assume that Ω has at least five vertices.

Claim 2. For each vertex v , $\Omega - v$ is unbalanced and has a balancing vertex.

Proof of Claim: Minimality implies that for any vertex v in Ω , $\Omega - v$ is not tangled. Since $\Omega - v$ is unbalanced and has no two disjoint unbalanced cycles, it must have a balancing vertex. ♣

Given an edge e with endpoints x and y , we denote the vertex in Ω/e resulting from the identification of x and y by v_e or v_{xy} .

Claim 3. For each edge e , Ω/e has v_e as its unique balancing vertex.

Proof of Claim: By minimality, Ω/e is not tangled and has no two vertex disjoint unbalanced cycles and so must therefore have a balancing vertex. If this balancing vertex is $u \neq v_e$, then every unbalanced cycle of Ω/e passes through u and so every unbalanced cycle of Ω passes through u which implies that u is a balancing vertex of Ω , a contradiction. ♣

Claim 4. Ω does not have a vertical 2-separation (A, B) in which B is balanced.

Proof of Claim: Suppose, for a contradiction, that (A, B) is a vertical 2-separation in which B is balanced. Let $\{x, y\} = V(A) \cap V(B)$, and let e be an edge in B not incident to at least one of x and y . By Claim 3, Ω/e has balancing vertex v_e . By our choice of e , $(A, B \setminus e)$ is a 2-separation of Ω/e , and $V(A) \cap V(B \setminus e)$ is either $\{x, y\}$, $\{x, v_e\}$ or $\{v_e, y\}$. In any case, since $B \setminus e$ is balanced, every unbalanced cycle of Ω/e either does not intersect $B \setminus e$ or intersects $B \setminus e$ in the edges of a path linking the two vertices of $V(A) \cap V(B \setminus e)$. But this implies that there is $v \in \{x, y\}$ such that every unbalanced cycle in Ω contains v . This makes v a balancing vertex of Ω , a contradiction. ♣

Claim 5. Ω is vertically 3-connected.

Proof of Claim: Suppose that Ω has a vertical 2-separation (A, B) and let $\{x, y\} = V(A) \cap V(B)$. By Claim 4, neither A nor B is balanced. Since Ω does not have a balancing vertex both $\Omega - x$ and $\Omega - y$ are unbalanced. Let C_x be an unbalanced cycle in $\Omega - x$ and C_y be an unbalanced cycle in $\Omega - y$. Without loss of generality $E(C_x) \subseteq A$ and either $E(C_y) \subseteq A$ or $E(C_y) \subseteq B$. It cannot be that $E(C_y) \subseteq B$ because then C_x and C_y would be vertex disjoint, a contradiction. Hence $E(C_x) \cup E(C_y) \subseteq A$; however, since B is unbalanced any unbalanced cycle C' in $\Omega:B$ intersects both vertices x and y and so $C_x \cup C_y \cup C'$ contains a biased $2C_3$ having no balanced 2-cycle, a contradiction. ♣

Now let $e = xy$ be an edge of Ω and let E_1, \dots, E_m be the unbalancing classes of edges incident to balancing vertex v_{xy} in Ω/e . We must have that $m \geq 2$. Now let $E_{x,i}$ be the edges of E_i that are incident to x in Ω and $E_{y,i}$ be the edges of E_i that are incident to y in Ω . Since $\Omega - y$ is unbalanced, at least two of $E_{x,1}, \dots, E_{x,m}$ are nonempty; similarly, at least two of $E_{y,1}, \dots, E_{y,m}$ must be nonempty. Let X be the set of vertices in $\Omega - y$ adjacent to x , and let Y be the set of vertices in $\Omega - x$ that are adjacent to y . Since the underlying graph of Ω is simple, $|X| \geq 2$ and $|Y| \geq 2$. Now take $x_1, x_2 \in X$ such that the xx_1 - and xx_2 -edges are in different sets $E_{x,1}, \dots, E_{x,m}$, and take two similarly defined $y_1, y_2 \in Y$. Since the underlying graph of Ω is simple, $x_1 \neq x_2$ and $y_1 \neq y_2$.

Claim 6. Vertices x_1, x_2, y_1, y_2 cannot be chosen so that $\{x_1, x_2\} \cap \{y_1, y_2\} = \emptyset$.

Proof of Claim: Since Ω is vertically 3-connected, there is a x_1x_2 -path P in $\Omega - \{x, y\}$. For $i \in \{1, 2\}$ let e_i denote the yy_i -edge in Ω . Since v_{yy_i} is a balancing vertex in Ω/e_i (by Claim 3), the path P must intersect y_1 and y_2 and so there is a y_1y_2 -path P' properly contained in P and P' that avoids both x_1 and x_2 . The unbalanced cycle C formed by y, y_1, P', y_2, y avoids x_1, x_2 , and x ; however, this yields a contradiction because contracting the xx_1 -edge in Ω leaves a biased graph with balancing vertex v_{xx_1} . ♣

Claim 7. Vertices x_1, x_2, y_1, y_2 cannot be chosen so that $|\{x_1, x_2\} \cap \{y_1, y_2\}| = 1$.

Proof of Claim: By way of contradiction assume that $|\{x_1, x_2\} \cap \{y_1, y_2\}| = 1$ where, without loss of generality, $x_2 = y_1$. As in the proof of Claim 6, any x_1x_2 -path P in $\Omega - \{x, y\}$ must intersect y_2 . Thus there is a y_1y_2 -path P' properly contained in P and avoiding x_1 , a contradiction. ♣

By Claims 6 and 7, every choice of x_1, x_2, y_1, y_2 has (without loss of generality) $x_1 = y_1$ and $x_2 = y_2$. This implies that $X = \{x_1, x_2\} = Y = \{y_1, y_2\}$, since otherwise we may choose a third vertex in X or Y so that $|\{x_1, x_2\} \cap \{y_1, y_2\}| \in \{0, 1\}$. Let e' and e'' be respectively the xx_1 - and xx_2 -edges in Ω and let f' and f'' be the yx_1 - and yx_2 -edges in Ω . It cannot be that e' and f' are in the same unbalancing equivalence class $E_j \in \{E_1, \dots, E_m\}$ because then $\Omega - x_2$ would be balanced. Similarly e'' and f'' are not in the same equivalence class. Because Ω is vertically 3-connected, there is an x_1x_2 -path P in $\Omega - \{x, y\}$. The subgraph of Ω on edges $E(P) \cup \{e, e', e'', f', f''\}$ is a subdivision of K_4 without a balanced triangle, a contradiction. □

As is often the case, graph minors are harder to work with than subgraphs. Theorem 3.4 is an analogue of Theorem 3.3 for topological subgraphs. The biased graph $T'_{2,3}$ of Figure 5 has exactly two triangles in it. The set of balanced cycles of $T'_{2,3}$ consists of exactly these two triangles. Let L denote the set of three links connecting the two triangles in $T'_{2,3}$. Note that $T'_{2,3}/L \cong T'_2$. For $i \in \{1, 2\}$, let $T'_{2,3-i}$ be the biased graph obtained from $T'_{2,3}$ by contracting i links from L .

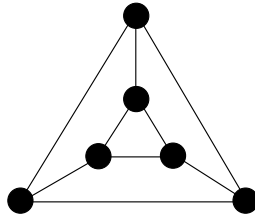


Figure 5: The biased graph $T'_{2,3}$ depicted has exactly two triangles, which comprise its set of balanced cycles.

Theorem 3.4. *If Ω is a vertically 2-connected and properly unbalanced biased graph, then Ω contains as a subgraph a subdivision of a base biased graph or a subdivision of a member of $\{T'_{2,3}, T'_{2,2}, T'_{2,1}\}$.*

Proof. If Ω is not tangled, then our result follows from Proposition 3.2. So assume that Ω is tangled. By Theorem 3.3, Ω contains a link minor (G, \mathcal{B}) that is either a biased K_4 without a balanced triangle or a biased $2C_3$ without a balanced 2-cycle. In the case of a biased K_4 , Ω

must contain as a subgraph a subdivision of (G, \mathcal{B}) because K_4 is a 3-regular graph. In the case of a biased $2C_3$, either Ω contains as a subgraph a subdivision of (G, \mathcal{B}) or Ω contains as a link minor (G', \mathcal{B}') which is vertically 2-connected, has minimum degree 3, and contains an edge e for which $(G', \mathcal{B}')/e = (G, \mathcal{B})$. Since Ω is tangled, the underlying graph of (G', \mathcal{B}') is as shown in Figure 7.

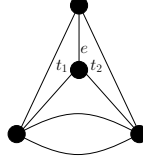


Figure 6

Figure 7: The graph G'

The reader may check that if $(G, \mathcal{B}) \not\cong T'_2$, then (G', \mathcal{B}') must contain as a subgraph a member of $\{D_{0,0}, D_{0,1}, D_{0,2}, D_{0,3}\}$ and therefore we have our desired subgraph of Ω . Now suppose that $(G, \mathcal{B}) \cong T'_2$. So (G', \mathcal{B}') contains exactly two balanced cycles that are both triangles or both quadrilaterals. In the latter case, we again have an element of $\{D_{0,0}, D_{0,1}, D_{0,2}, D_{0,3}\}$ as a subgraph of (G', \mathcal{B}') and we are done. In the former case, either Ω contains as a subgraph a subdivision of (G', \mathcal{B}') (which is isomorphic to $T'_{2,1}$) or Ω contains as a link minor (G'', \mathcal{B}'') which has minimum degree 3 and has an edge e_2 for which $(G'', \mathcal{B}'')/e_2 = (G', \mathcal{B}')$. Using similar reasoning either (G'', \mathcal{B}'') contains a subgraph that is a subdivision of an element of $\{D_{0,0}, D_{0,1}, D_{0,2}, D_{0,3}\}$ (in which case we are done) or (G'', \mathcal{B}'') is isomorphic to $T'_{2,2}$. Thus Ω contains a subdivision of (G'', \mathcal{B}'') as a subgraph (in which case we are done) or Ω contains a subdivision of (G''', \mathcal{B}''') where G is isomorphic to $T'_{2,3}$ or contains a subgraph that is a subdivision of an element of $\{D_{0,0}, D_{0,1}, D_{0,2}, D_{0,3}\}$. This completes our proof. \square

Biased graphs representing $U_{2,4}$. Two additional important biased graphs are U_2 and U_3 , shown in Figure 8; neither contains a balanced cycle. Note that $F(U_2) \cong F(U_3) \cong U_{2,4}$ and $L(U_3) \cong U_{2,4}$.



Figure 8: Two contrabalanced biased graphs representing $U_{2,4}$.

Our next theorem says that switching inequivalence of gain functions on a biased graph may always be found in a small minor, typically on one of our base graphs.

Theorem 3.5. *Let (G, \mathcal{B}) be a vertically 2-connected, loopless, and properly unbalanced biased graph and let Γ be an abelian group. If φ and ψ are switching inequivalent Γ -realizations of (G, \mathcal{B}) , then one of the following holds.*

- (G, \mathcal{B}) has a link minor (H, \mathcal{S}) that is a member of \mathcal{G}_0 and $\varphi|_H$ and $\psi|_H$ are switching inequivalent.
- (G, \mathcal{B}) contains a link minor $(H_3, \mathcal{S}_3) \cong U_3$ such that $\varphi|_{H_3}$ and $\psi|_{H_3}$ are switching inequivalent on the theta subgraph of U_3 and also (G, \mathcal{B}) contains a minor $(H_2, \mathcal{S}_2) \cong U_2$ such that $\varphi|_{H_2}$ and $\psi|_{H_2}$ are switching inequivalent on the 2-cycle of U_2 . Furthermore, (G, \mathcal{B}) is not tangled when this possibility holds.

Furthermore, if $\Gamma = \mathbb{F}^+$ and φ and ψ are inequivalent up to switching and scaling, then the same conclusions hold up to switching and scaling.

Proof. By Theorem 3.4, (G, \mathcal{B}) has a subgraph (G_0, \mathcal{B}_0) that is a subdivision of a member of $\mathcal{G}_0 \cup \{T'_{2,1}, T'_{2,2}, T'_{2,3}\}$. Since (G, \mathcal{B}) is vertically 2-connected and loopless, there is a sequence of vertically 2-connected subgraphs $(G_0, \mathcal{B}_0), \dots, (G_n, \mathcal{B}_n)$ such that $(G_n, \mathcal{B}_n) = (G, \mathcal{B})$ and $(G_{i+1}, \mathcal{B}_{i+1}) = (G_i, \mathcal{B}_i) \cup P_i$ for some path P_i that is internally disjoint from G_i . Let φ_i and ψ_i be the Γ -gain functions induced by φ and ψ on G_i . If φ_0 and ψ_0 are switching inequivalent (or switching and scaling for $\Gamma = \mathbb{F}^+$), then our result follows by Proposition 2.3. Otherwise, there is $t \in \{0, \dots, n-1\}$ such that φ_i and ψ_i are switching equivalent for $i \leq t$ and φ_{t+1} and ψ_{t+1} are switching inequivalent (or inequivalent up to switching and scaling). Let e be an edge on path P_t . Since G_{t+1} is vertically 2-connected, there is a spanning tree T_{t+1} of G_{t+1} not containing e . Let T_t be T_{t+1} restricted to G_t and so T_t is a spanning tree of G_t . Normalize φ_{t+1} and ψ_{t+1} on T_{t+1} which implies that φ_t and ψ_t are normalized on T_t as well. Since φ_t and ψ_t are switching equivalent and φ_{t+1} and ψ_{t+1} are switching inequivalent, $\varphi_t = \psi_t$ (by Proposition 2.2) while φ_{t+1} and ψ_{t+1} are equal everywhere aside from edge e . Hence for every cycle C of G_{t+1} containing path P_t , $\varphi_{t+1}(C) \neq \psi_{t+1}(C)$. Since φ_{t+1} and ψ_{t+1} are Γ -realizations of $(G_{t+1}, \mathcal{B}_{t+1})$, it must be that every such cycle C is unbalanced. Extend P_t to a path P that is internally disjoint from G_0 but whose endpoints are both on G_0 . Let φ' and ψ' be φ_{t+1} and ψ_{t+1} restricted to the biased graph $(G_0, \mathcal{B}_0) \cup P$. Again, φ' and ψ' are equal on every edge of $(G_0, \mathcal{B}_0) \cup P$ save for the edge e where they are unequal and every cycle C in $(G_0, \mathcal{B}_0) \cup P$ containing e is unbalanced. Now, in $(G_0, \mathcal{B}_0) \cup P$ there is a nonseparable link minor $(\widehat{G}, \widehat{\mathcal{B}}) = ((G_0, \mathcal{B}_0) \cup P)/K \setminus D$ for which $(\widehat{G}, \widehat{\mathcal{B}}) \setminus e$ is a base biased graph or $(\widehat{G}, \widehat{\mathcal{B}}) \setminus e/f$ is a base biased graph for some link f . The possibilities for $(\widehat{G}, \widehat{\mathcal{B}})$ are as shown in Figure 9.

Normalizing φ' and ψ' on a spanning tree of $(G_0, \mathcal{B}_0) \cup P$ that contains the contraction set K and let $\widehat{\varphi}$ and $\widehat{\psi}$ be the induced gain functions on $(\widehat{G}, \widehat{\mathcal{B}})$. Again, $\widehat{\varphi}$ and $\widehat{\psi}$ are equal on each edge aside from e where they are unequal and that every cycle through e is unbalanced. The first conclusion of our theorem holds in four of the six possibilities of Figure 9 and the second conclusion of our theorem holds in possibilities (iii) and (vi) which do not occur when (G, \mathcal{B}) is tangled. (Note contraction of an unbalanced loop is only necessary in possibility (vi).) \square

4 Representations of matroids of our base biased graphs

In this section we examine the relationship between gain functions on our base biased graphs and matrix representations of their associated frame and lift matroids. In Section 4.1 we

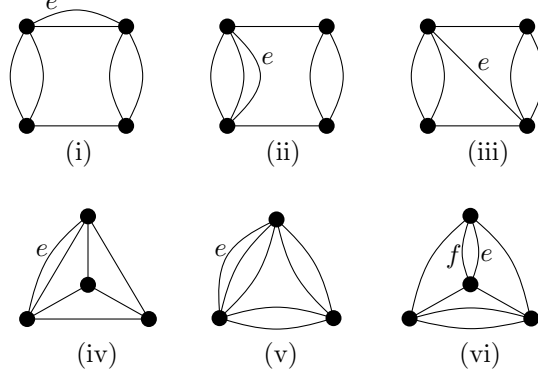


Figure 9: The possibilities for $(\widehat{G}, \widehat{\mathcal{B}})$.

show that canonical representations are projectively equivalent if and only if the associated gain functions are switching equivalent or switching-and-scaling equivalent. In Section 4.2 we show that any \mathbb{F} -representation of one of these matroids is projectively equivalent to a canonical \mathbb{F} -representation specific to the given graph.

4.1 Switching and projective equivalence

Given switching equivalent gain functions on a graph (with scaling for additive gain functions), it is easy to see that their corresponding canonical matrix representations are projectively equivalent. We give a proof of Proposition 4.1 for reference to certain ideas contained in it.

Proposition 4.1 (Zaslavsky [16, Sections. 2.3 and 4.3]). *Let G be a graph, and let φ and ψ be either \mathbb{F}^\times - or \mathbb{F}^+ -gain functions on G . If φ and ψ are switching equivalent (with scaling for \mathbb{F}^+), then their canonical matrix representations are projectively equivalent.*

Proof. Suppose φ and ψ are \mathbb{F}^\times -gain functions on G with $\varphi^\eta = \psi$. Let $V(G) = \{v_1, v_2, \dots, v_{|V(G)|}\}$. Let T be the diagonal matrix with rows and columns indexed by $V(G)$ in which diagonal entry T_{ii} is $\eta(v_i)$, and let S be the $|E(G)| \times |E(G)|$ diagonal matrix with diagonal entries $S_{jj} = \eta(v_i)^{-1}$ if vertex v_i is the tail of edge e_j . Then $TA_F(G, \varphi)S = A_F(G, \psi)$.

Now suppose φ and ψ are \mathbb{F}^+ -gain functions and there is a scalar $s \in \mathbb{F}^\times$ so that $s\varphi^\eta = \psi$. Let T be the $(n+1) \times (n+1)$ matrix whose first row is $[s \ \eta(v_1) \ \eta(v_2) \ \dots \ \eta(v_n)]$, first column is $[s \ 0 \ 0 \ \dots \ 0]^T$, and with the $n \times n$ identity matrix as the submatrix of the remaining rows and columns. Let S be the diagonal matrix with $s_{11} = 1/s$ and all other $s_{ii} = 1$. Then $TA_L(G, \varphi)S = A_L(G, \psi)$. \square

There is a natural condition arising from the proof of Proposition 4.1 for a certain converse statement to hold.

Proposition 4.2. *Let $A = A_F(G, \varphi)$ and $B = A_F(G, \psi)$ be projectively equivalent frame matrices. Then φ and ψ are switching equivalent if and only if $B = TAS$ where T and S are both diagonal and nonsingular.*

Proof. If φ and ψ are switching equivalent, then the proof of Proposition 4.1 shows that A and B are equivalent via diagonal and nonsingular matrices T and S . Conversely, suppose that $B = TAS$ where T and S are both diagonal and nonsingular. Since T is diagonal, row i of TA is obtained by multiplying row v_i of A by T_{ii} . Since both A and B are canonical frame representations, both have 1 in position v_i of column e_j whenever vertex v_i is the tail of edge e_j . Hence the diagonal elements S_{jj} of S satisfy $S_{jj} = T_{ii}^{-1}$, where v_i is the tail of e_j . Thus φ and ψ are switching equivalent via $\eta(v_i) = T_{ii}$ for each $v_i \in V(G)$. \square

4.1.1 Biased $2C_3$'s

Lemma 4.3. *Let $\varphi: \vec{E}(2C_3) \rightarrow \mathbb{F}^\times$ be a gain function with \mathcal{B}_φ containing no 2-cycle, and let $\psi: \vec{E}(2C_3) \rightarrow \mathbb{F}^\times$ be another gain function. Then φ and ψ are switching equivalent if and only if $A_F(2C_3, \varphi)$ and $A_F(2C_3, \psi)$ are projectively equivalent.*

Proof. If φ and ψ are switching equivalent, then $A_F(2C_3, \varphi)$ and $A_F(2C_3, \psi)$ are projectively equivalent by Proposition 4.1. To prove the converse, consider $A = A_F(2C_3, \varphi)$ and $B = A_F(2C_3, \psi)$ that are projectively equivalent. By normalizing on the spanning tree with edge set $\{e_1, e_3\}$, we may assume that φ labels $\vec{E}(2C_3)$ as shown in Figure 10.

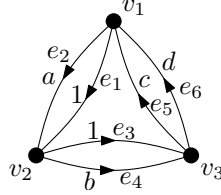


Figure 10: Gain function φ on $2C_3$

Since \mathcal{B}_φ contains no 2-cycle, $a \neq 1 \neq b$ and $c \neq d$. Since A and B are projectively equivalent, there is a nonsingular matrix T and a diagonal matrix S so that $TA = BS$. Denote by t_i row i of T , and by e_j column j of A . We have

$$A = \begin{matrix} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \end{matrix} & \begin{bmatrix} 1 & 1 & 0 & 0 & -c & -d \\ -1 & -a & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -b & 1 & 1 \end{bmatrix} \end{matrix}. \quad (4.1)$$

Entry $(BS)_{ij} = 0$ if and only if entry $(TA)_{ij} = 0$; consider the dot products $t_i \cdot e_j = 0$, where $(i, j) \in \{(3, 1), (3, 2), (1, 3), (1, 4), (2, 5), (2, 6)\}$. The product $t_3 \cdot e_1 = 0$ implies $T_{31} = T_{32}$, and $t_3 \cdot e_2 = 0$ implies $T_{31} = aT_{32}$. Together these imply (since $a \neq 1$) that $T_{32} = T_{31} = 0$. Similarly, $t_1 \cdot e_3 = t_1 \cdot e_4 = 0$ imply $T_{13} = T_{12} = 0$, and $t_2 \cdot e_5 = t_2 \cdot e_6 = 0$ imply $T_{21} = T_{23} = 0$. Hence T is diagonal, and so by Proposition 4.2, φ and ψ are switching equivalent. \square

Lemma 4.4. *Let $\varphi: \vec{E}(2C_3) \rightarrow \mathbb{F}^+$ be a gain function such that \mathcal{B}_φ contains no 2-cycle and let $\psi: \vec{E}(2C_3) \rightarrow \mathbb{F}^+$ be another gain function. Then φ and ψ are equivalent up to switching and scaling if and only if $A_L(2C_3, \varphi)$ and $A_L(2C_3, \psi)$ are projectively equivalent.*

Proof. If φ and ψ are switching-and-scaling equivalent, then their associated lift matrices are projectively equivalent by Proposition 4.1. Conversely, consider $A=A_L(2C_3, \varphi)$ and $B=A_L(2C_3, \psi)$ that are projectively equivalent. Normalizing on spanning tree $\{e_1, e_2\}$, and scaling if necessary, by Proposition 4.1 we may assume φ labels $2C_3$ as shown in Figure 11.

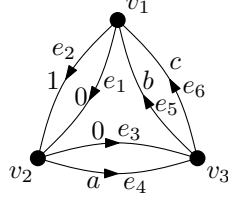


Figure 11: Gain function $\varphi: \vec{E}(2C_3) \rightarrow \mathbb{F}^+$.

Since φ has no balanced cycles of length 2, $a \neq 0$ and $b \neq c$. Lift matrix $A=A_L(2C_3, \varphi)$ is

$$A = \begin{array}{c} \text{gains} \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \end{matrix} \end{array} \begin{bmatrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ 0 & 1 & 0 & a & b & c \\ 1 & 1 & 0 & 0 & -1 & -1 \\ -1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 1 \end{bmatrix}. \quad (4.2)$$

There is a non-singular matrix T and a diagonal matrix S so that $TA = BS$. As with φ , by switching and scaling we may assume ψ also labels $\vec{E}(2C_3)$ as in Figure 11, replacing a , b , and c with x , y , and z , respectively. Then, denoting elements S_{ii} of S by s_i , we have

$$BS = \begin{bmatrix} 0 & s_2 & 0 & s_4x & s_5y & s_6z \\ s_1 & s_2 & 0 & 0 & -s_5 & -s_6 \\ -s_1 & -s_2 & s_3 & s_4 & 0 & 0 \\ 0 & 0 & -s_3 & -s_4 & s_5 & s_6 \end{bmatrix}.$$

This gives us 24 relations among the members of T , one for each dot product $t_i \cdot e_j$, where t_i is the i th column of T and e_j is the j th column of A . The eight relations $t_i \cdot e_j = 0$ yield $T_{12} = T_{13} = T_{14}$, $T_{21} = T_{31} = T_{41} = 0$, $T_{23} = T_{24}$, and $T_{42} = T_{43}$. Now, after establishing these relations, $t_3 \cdot e_5 = 0$ yields $T_{32} = T_{34}$ and so we have

$$T = \begin{bmatrix} T_{11} & T_{12} & T_{12} & T_{12} \\ 0 & T_{22} & T_{23} & T_{23} \\ 0 & T_{32} & T_{33} & T_{32} \\ 0 & T_{42} & T_{42} & T_{44} \end{bmatrix}.$$

Now the relations $s_1 = t_2 \cdot e_1$, $s_2 = t_2 \cdot e_2$, $s_3 = t_3 \cdot e_3$, $s_4 = t_3 \cdot e_4$, $s_5 = t_4 \cdot e_5$, $s_6 = t_4 \cdot e_6$, $-s_2 = t_3 \cdot e_2$, and $-s_3 = t_4 \cdot e_3$ yield $s_1 = s_2 = s_3 = s_4 = s_5 = s_6$. After this the relation $s_2 = t_1 \cdot e_2$ yields $T_{11} = s_1$. Now the relations $t_1 \cdot e_4 = s_1x$, $t_1 \cdot e_5 = s_1y$, and $t_1 \cdot e_6 = s_1z$ yield $a = x$, $b = y$ and $c = z$ which implies that φ and ψ are switching equivalent after scaling. \square

Lemma 4.5. *Suppose $\varphi: \vec{E}(2C_3) \rightarrow \mathbb{F}^\times$ and $\psi: \vec{E}(2C_3) \rightarrow \mathbb{F}^+$ are gain functions on $2C_3$, neither of which has a balanced 2-cycle. Then $A_F(2C_3, \varphi)$ and $A_L(2C_3, \psi)$ are not projectively equivalent.*

Proof. As in previous cases, without loss of generality we may assume that φ labels $2C_3$ as in Figure 10, and ψ as in Figure 11, replacing a with x , b with y , and c with z . Let $A = A_F(2C_3, \varphi)$ which is matrix 4.1. Let B be $A_L(2C_3, \psi)$ which is matrix 4.2 with the bottom row removed. Now suppose for a contradiction that there exists a non-singular matrix T and a diagonal matrix S so that $TA = BS$. Writing $S_{ii} = s_i$, and denoting row i of T by t_i and column j of A by e_j , we have $t_2 \cdot e_1 = s_1$, $t_2 \cdot e_2 = s_2$, $t_2 \cdot e_3 = 0$, and $t_2 \cdot e_4 = 0$. Together these imply that $T_{22} = T_{23} = 0$ and that $T_{21} = s_1 = s_2$. Moreover, we have $t_3 \cdot e_1 = -s_1$, $t_3 \cdot e_2 = -s_2$, $t_3 \cdot e_5 = 0$, and $t_3 \cdot e_6 = 0$. Since $s_1 = s_2$, $a \neq 0, 1$, and $c \neq d$, these imply that $T_{31} = T_{32} = T_{33} = 0$ which implies that T is singular, a contradiction. \square

4.1.2 Biased K_4 's

Lemma 4.6. *Let $\varphi: \vec{E}(K_4) \rightarrow \mathbb{F}^\times$ be a gain function with \mathcal{B}_φ containing no 3-cycle, and let $\psi: \vec{E}(K_4) \rightarrow \mathbb{F}^\times$ be another gain function. Then φ and ψ are switching equivalent if and only if $A_F(K_4, \varphi)$ and $A_F(K_4, \psi)$ are projectively equivalent.*

Proof. By switching we may assume that φ and ψ are both equal to the identity on a $K_{1,3}$ -subgraph Y . This allows us to consider φ and ψ as gain functions on $\nabla_Y K_4 = 2C_3$. Now $\nabla_Y(K_4, \mathcal{B}_\varphi) = (2C_3, \mathcal{B}_\varphi)$ and $\nabla_Y(K_4, \mathcal{B}_\psi) = (2C_3, \mathcal{B}_\psi)$. Since φ has no balanced triangles in K_4 , neither has it any balanced 2-cycles in $2C_3$. So Lemma 4.3 implies that φ and ψ are switching equivalent if and only if $A_F(2C_3, \varphi)$ and $A_F(2C_3, \psi)$ are projectively equivalent and so Propositions 2.16 and 2.15 imply that φ and ψ are switching equivalent if and only if $A_F(K_4, \varphi)$ and $A_F(K_4, \psi)$ are projectively equivalent. \square

Again using $Y\Delta$ -exchanges as in the proof of Lemma 4.6 we get Lemmas 4.7 and 4.8.

Lemma 4.7. *Let $\varphi: \vec{E}(K_4) \rightarrow \mathbb{F}^+$ be a gain function with \mathcal{B}_φ containing no 3-cycle and let $\psi: \vec{E}(K_4) \rightarrow \mathbb{F}^+$ be another gain function. Then φ and ψ are switching-and-scalaing equivalent if and only if $A_L(K_4, \varphi)$ and $A_L(K_4, \psi)$ are projectively equivalent.*

Lemma 4.8. *Let (K_4, \mathcal{B}) be a biased graph with \mathcal{B} containing no balanced triangle, and let $\varphi: \vec{E}(K_4) \rightarrow \mathbb{F}^\times$ and $\psi: \vec{E}(K_4) \rightarrow \mathbb{F}^+$ be gain functions on K_4 realizing \mathcal{B} . Then $A = A_F(K_4, \varphi)$ and $B = A_L(K_4, \psi)$ are projectively inequivalent.*

4.1.3 Biased $2C_4''$'s

Lemma 4.9. *Let $\varphi: \vec{E}(2C_4'') \rightarrow \mathbb{F}^\times$ be a gain function such that \mathcal{B}_φ has no balanced 2-cycle and let $\psi: \vec{E}(2C_4'') \rightarrow \mathbb{F}^\times$ be another gain function. Then $A_F(2C_4'', \varphi)$ and $A_F(2C_4'', \psi)$ are projectively equivalent if and only if φ and ψ are switching equivalent.*

Proof. Let $A = A_F(2C_4'', \varphi)$ and $B = A_F(2C_4'', \psi)$. If φ and ψ are switching equivalent, then A and B are projectively equivalent by Proposition 4.1. To prove the converse, let T and S be matrices with $TA = BS$ (where T is nonsingular and S is a diagonal matrix scaling the

columns of B). We may assume without loss of generality that the edge orientations chosen to define B are the same as those chosen to define A ; by normalizing on the spanning tree with edges e_3, e_4, e_5 , we may assume without loss of generality that φ labels $\vec{E}(K_4)$ as shown in Figure 12

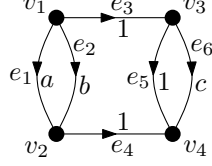


Figure 12: Labeled $2C_4''$ with a normalized gain function.

where $a \neq b$ and $c \neq 1$. Thus

$$A = \begin{matrix} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -a & -b & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & -1 & -c \end{bmatrix} \end{matrix}.$$

Now, each entry B_{ij} of B is zero if and only if entry $(TA)_{ij} = 0$. Now for a fixed row i , there are three distinct j such that $t_i \cdot e_j = 0$. The reader can check, for each i , that these three relations yield $T_{ij} \neq 0$ iff $i = j$. For example $0 = t_1 \cdot e_4$ implies $T_{12} = T_{14}$, $0 = t_1 \cdot e_5$ implies $T_{13} = T_{14}$, and $0 = t_1 \cdot e_6$ implies $T_{13} = cT_{14}$. Since $c \neq 1$ we then get that $T_{12} = T_{13} = T_{14} = 0$. Thus T is a diagonal matrix and so Proposition 4.2 implies that φ and ψ are switching equivalent. \square

Lemma 4.10. *Let $\varphi: \vec{E}(2C_4'') \rightarrow \mathbb{F}^+$ be a gain function which yields no balanced 2-cycle and let $\psi: \vec{E}(2C_4'') \rightarrow \mathbb{F}^+$ be another gain functions. Then $A_L(2C_4'', \varphi)$ and $A_L(2C_4'', \psi)$ are projectively equivalent if and only if φ and ψ are equivalent by switching and scaling.*

Proof. The easy direction again follows from Proposition 4.1. For the converse, without loss of generality assume that $\varphi(e_1) = \psi(e_1) = 1$, $\varphi(e_2) = a$, $\psi(e_2) = x$, $\varphi(e_3) = \psi(e_3) = 0$, $\varphi(e_4) = \psi(e_4) = 0$, $\varphi(e_5) = \psi(e_5) = 0$, $\varphi(e_6) = b$, and $\psi(e_6) = y$ (where $2C_4''$ has edges and orientations as in Figure 12) such that neither a nor x is 1 and neither b nor y is 0. Let $A = A_L(2C_4'', \varphi)$ and $B = A_L(2C_4'', \psi)$, and let T and S be matrices with $TA = BS$, where S is diagonal (with $s_i = S_{ii}$) scaling the columns of B . Denoting row i of T by t_i and column j of A by e_j we have $t_i \cdot e_j = 0$ for 15 pairs (i, j) , three pairs for any fixed row t_i . Given the fact that $a \neq 1$ and $b \neq 0$, the reader can check that these 15 relations yield

$$T = \begin{bmatrix} T_{11} & T_{12} & T_{12} & T_{12} & T_{12} \\ 0 & T_{22} & T_{23} & T_{23} & T_{23} \\ 0 & T_{32} & T_{33} & T_{32} & T_{32} \\ 0 & T_{42} & T_{42} & T_{44} & T_{42} \\ 0 & T_{52} & T_{52} & T_{52} & T_{55} \end{bmatrix}.$$

After this, each column e_j has $i, k \geq 2$ such that $t_i \cdot e_j = s_j$ and $t_k \cdot e_j = -s_j$. These 12 relations yield $s_1 = s_2 = s_3 = s_4 = s_5 = s_6$. The relation $t_1 \cdot e_1 = s_1$ yields $T_{11} = s_1$. Now the relation $t_1 \cdot e_2 = s_1 x$ yields $a = x$ and the relation $t_1 \cdot e_6 = s_1 y$ yields $b = y$. Thus A and B are equivalent by switching and scaling. \square

4.1.4 U_2 and U_3

Suppose U_2 and U_3 are labeled with edge orientations as in Figure 13. Denote the underlying graphs by U_2 and U_3 , respectively.

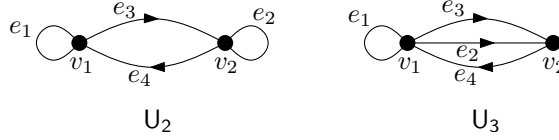


Figure 13

Lemma 4.11. *Let φ and ψ be \mathbb{F}^\times -realizations of U_2 . Then $A_F(U_2, \varphi)$ and $A_F(U_2, \psi)$ are projectively equivalent if and only if $\varphi(e_3 e_4) = \psi(e_3 e_4)$.*

Proof. The matrices $A_F(U_2, \varphi)$ and $A_F(U_2, \psi)$ are of the following form

$$\begin{bmatrix} 1 & 0 & 1 & -g \\ 0 & 1 & -1 & 1 \end{bmatrix}. \quad (4.3)$$

These are in standard form relative to the basis $\{e_1, e_2\}$ and so are projectively equivalent iff the entry g is the same for both $A_F(U_2, \varphi)$ and $A_F(U_2, \psi)$. The result follows. \square

Lemma 4.12. *Let φ and ψ be \mathbb{F}^+ -realizations of U_3 . Then $A_L(U_3, \varphi)$ and $A_L(U_3, \psi)$ are projectively equivalent if and only if $\varphi|_{\{e_2, e_3, e_4\}}$ and $\psi|_{\{e_2, e_3, e_4\}}$ are switching-and-scaling equivalent.*

Proof. If $A = A_L(U_3, \varphi)$ and $B = A_L(U_3, \psi)$ are projectively equivalent, then there is an invertible matrix T and diagonal matrix S such that $TA = BS$ (gain let $s_i = S_{ii}$). After switching and scaling, $\varphi(e_1) = \psi(e_1) = 1$, $\varphi(e_2) = \psi(e_2) = 0$, $\varphi(e_3) = \psi(e_3) = 1$, $\varphi(e_4) = a$, and $\psi(e_4) = x$. Thus

$$\begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & a \\ 0 & 1 & 1 & -1 \\ 0 & -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} s_1 & 0 & s_3 & s_4 x \\ 0 & s_2 & s_3 & -s_4 \\ 0 & -s_2 & -s_3 & s_4 \end{bmatrix}.$$

This yields $T_{11} = s_1$, $T_{21} = T_{31} = 0$, and $T_{12} = T_{13}$ and so

$$\begin{bmatrix} s_1 & T_{12} & T_{12} \\ 0 & T_{22} & T_{23} \\ 0 & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & a \\ 0 & 1 & 1 & -1 \\ 0 & -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} s_1 & 0 & s_3 & s_4 x \\ 0 & s_2 & s_3 & -s_4 \\ 0 & -s_2 & -s_3 & s_4 \end{bmatrix}$$

which yields $s_1 = s_2 = s_3 = s_4$. After this $a = x$ and so φ and ψ are switching-and-scaling equivalent. \square

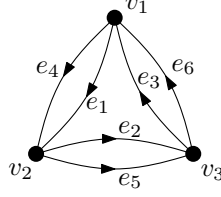


Figure 14

4.2 All \mathbb{F} -representations are canonical

In this section, we show that every \mathbb{F} -representation of a matroid of a base biased graph in \mathcal{G}_0 is projectively equivalent to a canonical representation.

4.2.1 Biased $2C_3$'s

Lemma 4.13. *If $(2C_3, \mathcal{B})$ is a labeled biased graph with no balanced 2-cycle, then every \mathbb{F} -representation of $F(2C_3, \mathcal{B}) = L(2C_3, \mathcal{B})$ is projectively equivalent to a unique canonical representation particular to the biased graph $(2C_3, \mathcal{B})$.*

Proof. We may assume that $2C_3$ is labelled and has edge orientations as shown in Figure 14. Let A be a matrix over \mathbb{F} representing $F(2C_3, \mathcal{B})$. By Proposition 3.1 we may assume that $\{e_1, e_2, e_3\}$ is a basis, and so that the first three columns of A represent e_1, e_2 , and e_3 , and that these columns form the identity matrix. Observe that the only form a 3-circuit may take in $2C_3$ is a balanced triangle. Now since neither e_5 nor e_6 forms a triangle with $\{e_2, e_3\}$, both the columns corresponding to e_5 and e_6 are nonzero in their first row. Hence we may assume (by applying elementary row operations and column scaling) that A is of the form

$$A = \begin{bmatrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & a & c \\ 0 & 0 & 1 & 1 & b & d \end{bmatrix}$$

for some $a, b, c, d \in \mathbb{F}$. The fact that the only form a 3-circuit may take in $2C_3$ is a balanced triangle further yields the following series of claims.

1. $b, c \neq 0$: If $b = 0$, then $\{e_1, e_2, e_5\}$ is a circuit. If $c = 0$, then $\{e_1, e_3, e_6\}$ is a circuit. These are both contradictions.
2. $a \neq b$: If so, then $a \neq 1$, as then e_4 and e_5 would form a parallel pair, a contradiction. But then e_1, e_4, e_5 form a circuit, also a contradiction.
3. $b \neq 1$: If so, then e_2, e_4, e_5 form a circuit, a contradiction.
4. $c \neq d$: If so, then certainly $c \neq 1$ as e_4 and e_6 are not a parallel pair. But then $\{e_1, e_4, e_6\}$ is a circuit, a contradiction.
5. $c \neq 1$: If so, $\{e_3, e_4, e_6\}$ is a circuit, a contradiction.

6. $a \neq c$: If so, then $d \neq b$ since e_5 and e_6 are not a parallel pair. But then $\{e_3, e_5, e_6\}$ is a circuit, a contradiction.
7. $b \neq d$: If so, then $a \neq c$ since e_5 and e_6 are not a parallel pair. But then $\{e_2, e_5, e_6\}$ is a circuit, a contradiction.

The matrix

$$T = \begin{bmatrix} 1 & 0 & -1/b \\ -1 & 1/c & 0 \\ 0 & -1/c & 1/c \end{bmatrix}$$

has determinant $1/c^2 - 1/bc$, which is nonzero as long as $b \neq c$. Moreover

$$TA = \begin{bmatrix} 1 & 0 & -\frac{1}{b} & \frac{b-1}{b} & 0 & \frac{b-d}{b} \\ -1 & \frac{1}{c} & 0 & \frac{1-c}{c} & \frac{a-c}{c} & 0 \\ 0 & -\frac{1}{c} & \frac{1}{c} & 0 & \frac{b-a}{c} & \frac{d-c}{c} \end{bmatrix}$$

which by claims 1-7 above has exactly two nonzero entries in each column. Scaling the columns of TA clearly yields a canonical frame matrix particular to $(2C_3, \mathcal{B})$.

If $b = c$, let

$$T = \begin{bmatrix} 0 & 0 & 1 \\ b & 0 & -1 \\ -b & 1 & 0 \end{bmatrix}$$

The determinant of T is b , so T is non-singular. Now

$$TA = \begin{bmatrix} 0 & 0 & 1 & 1 & b & d \\ -b & 0 & -1 & b-1 & 0 & b-d \\ b & 1 & 0 & 1-b & a-b & 0 \end{bmatrix}$$

By claims 1, 2, 3, and 7, none of entries given in terms of a , b , or d are zero. Hence after column scaling and appending a fourth row obtained by negating the sum of rows 2 and 3, TA is a lift matrix particular to $(2C_3, \mathcal{B})$, as required.

Uniqueness follows from Lemmas 4.3, 4.4, and 4.5. \square

Lemma 4.14. *If (G, \mathcal{B}) is a labeled biased graph isomorphic to an element of $\{\mathcal{T}'_{2,1}, \mathcal{T}'_{2,2}, \mathcal{T}'_{2,3}\}$, then every \mathbb{F} -representation of $F(G, \mathcal{B}) = L(G, \mathcal{B})$ is projectively equivalent to a unique canonical representation particular to the biased graph (G, \mathcal{B}) .*

Proof. The uniqueness follows from Proposition 2.3 and Lemma 4.3 so we need only show that an arbitrary \mathbb{F} -representation A of $F(G, \mathcal{B})$ is canonical.

First, consider $\mathcal{T}'_{2,1}$. If Y is a $K_{1,3}$ -subgraph of $\mathcal{T}'_{2,1}$, then $\nabla_Y \mathcal{T}'_{2,1}$ is obtained from \mathcal{T}'_2 by the addition of an edge, call it e , that creates a balanced 2-cycle. Hence e is a parallel to some other element of $F(\nabla_Y \mathcal{T}'_{2,1}) = \nabla_Y F(\mathcal{T}'_{2,1})$. This implies that \mathbb{F} -representation $\nabla_Y A$ of matroid $\nabla_Y F(\mathcal{T}'_{2,1})$ is projectively equivalent to a canonical representation by Lemma 4.13. Thus A is projectively equivalent to a canonical representation by Proposition 2.14.

Second, $\nabla_Y \mathcal{T}'_{2,2}$ is obtained from $\mathcal{T}'_{2,1}$ by the addition of an edge that creates a balanced 2-cycle. That A is projectively equivalent to a canonical matrix follows in an analogous fashion to the argument in the previous paragraph using $\mathcal{T}'_{2,1}$ rather than \mathcal{T}'_2 .

Last, $\nabla_Y \mathcal{T}'_{2,3}$ is obtained from $\mathcal{T}'_{2,2}$ by the addition of an edge that creates a balanced 2-cycle and our result follows as before. \square

4.2.2 Biased K_4 's

Lemma 4.15. *If (K_4, \mathcal{B}) is a labeled biased graph with no balanced 3-cycle, then every \mathbb{F} -representation of $F(K_4, \mathcal{B}) = L(K_4, \mathcal{B})$ is projectively equivalent to a unique canonical representation particular to the biased graph (K_4, \mathcal{B}) .*

Proof. Uniqueness follows from Lemma 4.6. Here $(K_4, \mathcal{B}) \cong D_{0,i}$ for some $i \in \{0, 1, 2, 3\}$. Letting Y be a $K_{1,3}$ -subgraph of (K_4, \mathcal{B}) we get $\nabla_Y(K_4, \mathcal{B}) \cong T_{i+1}$. If A is an \mathbb{F} -representation of $F(K_4, \mathcal{B}) = L(K_4, \mathcal{B})$, then $\nabla_Y A$ is an \mathbb{F} -representation of $\nabla_Y F(K_4, \mathcal{B}) = \nabla_Y L(K_4, \mathcal{B}) = F(\nabla_Y(K_4, \mathcal{B})) = L(\nabla_Y(K_4, \mathcal{B}))$. By Lemma 4.13, $\nabla_Y A$ is projectively equivalent to a canonical \mathbb{F} -representation and so A is projectively equivalent to a canonical \mathbb{F} -representation by Proposition 2.14. \square

4.2.3 Biased $2C_4''$'s

Lemma 4.16. *If $(2C_4'', \mathcal{B})$ is a labeled biased graph with no balanced 2-cycle, then every \mathbb{F} -representation of $F(2C_4'', \mathcal{B})$ is projectively equivalent to a unique canonical frame matrix that is particular to the biased graph $(2C_4'', \mathcal{B})$.*

Proof. The uniqueness aspect of representations follows from Lemma 4.9. We need only show that any given \mathbb{F} -representation of $F(2C_4'', \mathcal{B})$ is projectively equivalent to a frame matrix particular to the biased graph $(2C_4'', \mathcal{B})$. Assume without loss of generality that the labels on $2C_4''$ are as shown in Figure 15. There are three possibilities for \mathcal{B} : $|\mathcal{B}| \in \{0, 1, 2\}$.

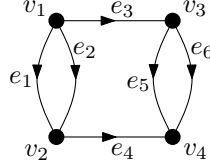


Figure 15

Assume first that $|\mathcal{B}|$ is 0 or 1; i.e. either $\mathcal{B} = \emptyset$ or, without loss of generality, $\mathcal{B} = \{e_1, e_3, e_4, e_6\}$. Then it is easy to see that A is projectively equivalent to

$$A' = \begin{bmatrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & a \\ 0 & 0 & 1 & 0 & 1 & b \\ 0 & 0 & 0 & 1 & 1 & c \end{bmatrix}$$

where $a, b, c \in \mathbb{F}$ are distinct, neither of b nor c is 0, and none of a , b , or c are 1. (If $\mathcal{B} = \emptyset$, then $a \neq 0$, and if $\mathcal{B} = \{e_1, e_3, e_4, e_5\}$ then $a = 0$.) Let

$$T = \begin{bmatrix} b-a & 1-b & a-1 & 0 \\ a-c & c-1 & 0 & 1-a \\ 0 & 0 & 1-a & 0 \\ 0 & 0 & 0 & a-1 \end{bmatrix}$$

Then $\det(T) = (a - 1)^3(c - b) \neq 0$, and

$$TA' = \begin{bmatrix} b-a & 1-b & a-1 & 0 & 0 & 0 \\ a-c & c-1 & 0 & 1-a & 0 & 0 \\ 0 & 0 & 1-a & 0 & 1-a & (1-a)b \\ 0 & 0 & 0 & a-1 & a-1 & (a-1)c \end{bmatrix}$$

which has the desired canonical form after column scaling, as required.

So assume now that $|\mathcal{B}| = 2$; without loss of generality that $\{\{e_1, e_3, e_4, e_6\}, \{e_2, e_3, e_4, e_5\}\}$. Then A is projectively equivalent to

$$A' = \begin{array}{c} \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & b \\ 0 & 0 & 0 & 1 & 1 & c \end{bmatrix} \end{array}$$

where b and c are nonzero, distinct, and not equal to 1. Let

$$T = \begin{bmatrix} -b & -1 & 1 & 0 \\ c & 1 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The determinant of T is $b - c \neq 0$, so T is nonsingular. Now

$$TA' = \begin{bmatrix} -b & -1 & 1 & 0 & 0 & 0 \\ c & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & -b \\ 0 & 0 & 0 & 1 & 1 & c \end{bmatrix}$$

which has the desired canonical form after column scaling, as required. \square

Lemma 4.17. *Let \mathbb{F} be a field and let A be a matrix over \mathbb{F} representing $L(2C_4'', \mathcal{B})$, where \mathcal{B} contains no 2-cycle. Then A is projectively equivalent to a unique canonical lift representation specific to the biased graph $(2C_4'', \mathcal{B})$.*

Proof. Again, let the edges of $2C_4''$ be labeled as in Figure 12. If $|\mathcal{B}| = 2$, then $L(2C_4'', \mathcal{B})$ is binary which makes A projectively unique and our result follows. So without loss of generality assume that $\mathcal{B} \subseteq \{\{e_1, e_3, e_4, e_6\}\}$. Since $\{e_1, e_2, e_5, e_6\}$ is a circuit, A is projectively equivalent to

$$A' = \begin{array}{c} \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & a \\ 0 & 0 & 1 & 0 & 1 & b \\ 0 & 0 & 0 & 1 & 1 & b \end{bmatrix} \end{array}$$

for some $a, b \in \mathbb{F}$. It is easy to check that a and b must be distinct, that neither a nor b is 1, that $b \neq 0$, and $a = 0$ iff $|\mathcal{B}| = 1$. Let

$$T = \begin{bmatrix} 0 & 1-b & 0 & 0 \\ b-a & 1-b & a-1 & 0 \\ a-b & b-1 & 0 & 1-a \\ 0 & 0 & 1-a & 0 \end{bmatrix}$$

Then $\det(T) = (a-1)^2(a-b)(b-1) \neq 0$, so T is nonsingular, and

$$TA' = \begin{bmatrix} 0 & 1-b & 0 & 0 & 1-b & a-ab \\ b-a & 1-b & a-1 & 0 & 0 & 0 \\ a-b & b-1 & 0 & 1-a & 0 & 0 \\ 0 & 0 & 1-a & 0 & 1-a & b-ab \end{bmatrix}$$

which, after column scaling and appending a fifth row obtained by negating the sum of rows 2, 3, and 4, is a canonical lift matrix particular to $(2C_4'', \mathcal{B})$, as desired. \square

4.2.4 Biased $2C_3 \setminus e$'s

We denote the graph obtained from $2C_3$ by deleting an edge by $2C_3 \setminus e$.

Lemma 4.18. *Let $(2C_3 \setminus e, \mathcal{B})$ be a biased graph with no balanced 2-cycles, and suppose A is an \mathbb{F} -representation of $F(2C_3 \setminus e, \mathcal{B}) = L(2C_3 \setminus e, \mathcal{B})$. Then A is projectively equivalent to a canonical frame matrix particular to $(2C_3 \setminus e, \mathcal{B})$ or a roll-up of $(2C_3 \setminus e, \mathcal{B})$, and A is projectively equivalent to a canonical lift matrix particular to $(2C_3 \setminus e, \mathcal{B})$.*

Proof. We may assume that \mathbb{F} is neither $GF(2)$ nor $GF(3)$, since in these cases A is projectively unique, so our result follows. Assume $2C_3 \setminus e$ is labeled as in Figure 16. Then $\{e_1, e_2, e_3\}$

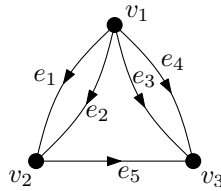


Figure 16: $2C_3 \setminus e$.

is a basis, so we may assume the first three columns of A are labelled e_1, e_2, e_3 , and that these columns form an identity matrix. Hence we may assume (consider fundamental circuits) that

$$A = \begin{bmatrix} e_1 & e_2 & e_3 & e_4 & e_5 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & a \\ 0 & 0 & 1 & 1 & b \end{bmatrix}$$

for some $a, b \in \mathbb{F}$, where $b \neq 0$. Note $\{e_3, e_4, e_5\}$ is not a circuit, so $a \neq 1$.

Suppose first that $a = 0$. Choose $x, y \in \mathbb{F}$ such that $x \neq 0$ and $y \notin \{0, -x\}$. Let

$$T = \begin{bmatrix} x & y & -x/b \\ -x & x & 0 \\ 0 & 0 & x/b \end{bmatrix}$$

Then $\det(T) = x^2(x + y)/b \neq 0$, so T is nonsingular, and

$$TA = \begin{bmatrix} x & y & \frac{-x}{b} & \frac{x(b-1)+by}{b} & 0 \\ -x & x & 0 & 0 & -x \\ 0 & 0 & \frac{x}{b} & \frac{x}{b} & x \end{bmatrix}$$

Each column of TA has at most two nonzero entries, so after column scaling TA is canonical frame particular to $(2C_3 \setminus e, \mathcal{B})$ or a rollup of $(2C_3 \setminus e, \mathcal{B})$. Now define

$$T = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & -1/b \\ -1 & 1 & 0 \end{bmatrix}$$

The determinant of T is $1/b$, so T is nonsingular, and

$$TA = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1/b & -1/b & 0 \\ -1 & 1 & 0 & 0 & -1 \end{bmatrix}$$

After appropriate column scaling and appending a fourth row equal to the negation of the sum of rows 2 and 3, TA is a canonical lift matrix particular to $(2C_3 \setminus e, \mathcal{B})$.

Now assume $a \neq 0$. Choose $x, y \in \mathbb{F}$ such that neither x nor y is 0 and $x/y \neq b/(a-1)$. Define

$$T = \begin{bmatrix} x & \frac{-x-by}{a} & y \\ -x & x & 0 \\ 0 & 0 & -y \end{bmatrix}$$

The determinant of T is $xy(x - ax + by)/a$, which by our choice of x and y is nonzero, and

$$TA = \begin{bmatrix} x & \frac{-x-by}{a} & y & \frac{x(a-1)+y(a-b)}{a} & 0 \\ -x & x & 0 & 0 & x(a-1) \\ 0 & 0 & -y & -y & -by \end{bmatrix}$$

which, after column scaling, is projectively equivalent to a canonical frame matrix particular to $(2C_2 \setminus e, \mathcal{B})$ or a rollup of $(2C_3 \setminus e, \mathcal{B})$. Finally, define

$$T = \begin{bmatrix} 0 & 1 & -a/b \\ 1 & -1 & (a-1)/b \\ -1 & 1 & 0 \end{bmatrix}$$

The determinant of T is $(1-a)/b$, and so nonzero, and

$$TA = \begin{bmatrix} 0 & 1 & -a/b & \frac{b-a}{b} & 0 \\ 1 & -1 & \frac{a-1}{b} & \frac{a-1}{b} & 0 \\ -1 & 1 & 0 & 0 & a-1 \end{bmatrix}$$

After appropriate column scaling and appending a fourth row equal to the negation of the sum of rows 2 and 3, we have a canonical lift matrix particular to $(2C_3 \setminus e, \mathcal{B})$. \square

5 Main results

We begin with our main result equating projective equivalence with switching equivalence. This answers Conjectures 2.8 and 4.8 of Zaslavsky from [16] in the affirmative: vertical 2-connectivity, no balancing vertex, and looplessness are all clearly necessary conditions for such a theorem.

Theorem 5.1 (Main Result II). *Let (G, \mathcal{B}) be a vertically 2-connected, loopless, and properly unbalanced biased graph.*

1. *If φ and ψ are \mathbb{F}^\times -realizations of (G, \mathcal{B}) , then $A_F(G, \varphi)$ and $A_F(G, \psi)$ are projectively equivalent iff φ and ψ are switching equivalent.*
2. *If φ and ψ are \mathbb{F}^+ -realizations of (G, \mathcal{B}) , then $A_L(G, \varphi)$ and $A_L(G, \psi)$ are projectively equivalent iff φ and ψ are switching-and-scaling equivalent.*
3. *If φ is an \mathbb{F}^\times -realizations of (G, \mathcal{B}) and ψ is an \mathbb{F}^+ -realizations of (G, \mathcal{B}) , then $A_F(G, \varphi)$ and $A_L(G, \psi)$ are not projectively equivalent.*

Proof. For Parts (1) and (2), the easy direction is by Proposition 4.1. For the converse of Part (1) assume that φ and ψ are \mathbb{F}^\times -realizations that are not switching equivalent. By Theorem 3.5, there is a minor (H, \mathcal{S}) of (G, \mathcal{B}) such that either (H, \mathcal{S}) is a base biased graph with $\varphi|_H$ and $\psi|_H$ switching inequivalent or $(H, \mathcal{S}) \cong U_2$ and $\varphi|_H$ and $\psi|_H$ are switching inequivalent on the 2-cycle. By Lemmas 4.3, 4.6, 4.9, and 4.11 $A_F(H, \varphi|_H)$ and $A_F(H, \psi|_H)$ are not projectively equivalent and so $A_F(G, \varphi)$ and $A_F(G, \psi)$ are not projectively equivalent. For the converse of Part (2), the proof is similar using inequivalence up to switching and scaling in Theorem 3.5 and using Lemmas 4.4, 4.7, 4.10, and 4.12.

For Part (3), we use Proposition 3.2 (if (G, \mathcal{B}) contains two vertex disjoint unbalanced cycles) or Theorem 3.5 (if (G, \mathcal{B}) is tangled) to get a \mathcal{G}_0 -minor (H, \mathcal{S}) of (G, \mathcal{B}) . The result follows by Lemmas 4.5 and 4.8 and the fact that the lift and frame matroids of a biased $2C_4''$ without a balanced 2-cycle are not equal. \square

Theorem 5.2 (Main Result III). *Let \mathbb{F} be a field and let (G, \mathcal{B}) be a vertically 2-connected and properly unbalanced biased graph.*

1. *If (G, \mathcal{B}) has two vertex-disjoint unbalanced cycles and A is an \mathbb{F} -matrix representing $F(G, \mathcal{B})$, then A is projectively equivalent to some $A_F(G, \varphi)$.*
2. *If (G, \mathcal{B}) has two vertex-disjoint unbalanced cycles and A is an \mathbb{F} -matrix representing $L(G, \mathcal{B})$, then A is projectively equivalent to some $A_L(G, \varphi)$.*
3. *If (G, \mathcal{B}) does not have two vertex-disjoint unbalanced cycles and A is an \mathbb{F} -matrix representing $F(G, \mathcal{B}) = L(G, \mathcal{B})$, then A is projectively equivalent to some $A_F(G, \varphi)$ or some $A_L(G, \psi)$ but not both.*

Furthermore, if (G, \mathcal{B}) has no loops, then the canonical representation particular to (G, \mathcal{B}) in each part is also unique.

Lemma 5.3. *Let Ω be a connected biased graph such that ℓ is an joint and $\Omega \setminus \ell$ is a biased K_4 without a balanced triangle or $2C_3$ without a balanced 2-cycle.*

1. *If $A_F(G \setminus \ell, \varphi)$ is a frame-matrix representation of $F(\Omega \setminus \ell) = L(\Omega \setminus \ell)$ over \mathbb{F} , then $A_F(G \setminus \ell, \varphi)$ does not extend to an \mathbb{F} -representation of $L(\Omega)$.*
2. *If $A_L(G \setminus \ell, \psi)$ is a lift-matrix representation of $F(\Omega \setminus \ell) = L(\Omega \setminus \ell)$ over \mathbb{F} , then $A_L(G \setminus \ell, \psi)$ does not extend to an \mathbb{F} -representation of $F(\Omega)$.*

Proof. We give a detailed proof for the case in which Ω is a biased $2C_3$. The case for which Ω is a biased K_4 then follows from ΔY - and $Y \Delta$ -exchanges by Propositions 2.13 and 2.14.

Part 1 Suppose by way of contradiction that \mathbb{F} -matrix A represents $L(\Omega)$ with column $\vec{\ell}$ corresponding to the element ℓ and $A \setminus \vec{\ell} = A_F(G \setminus \ell, \varphi)$. Since ℓ is not a loop of $L(\Omega)$, the column $\vec{\ell}$ is nonzero. If $\vec{\ell}$ has weight 1, then A is an \mathbb{F} -representation of a $F(\Omega')$ where Ω' is obtained from $\Omega \setminus \ell$ by attaching a joint to some vertex; however, we now get that $F(\Omega') \neq L(\Omega)$ by comparing circuits, a contradiction. If $\vec{\ell}$ has weight 2, then A is an \mathbb{F} -representation of $F(\Omega')$ in where Ω' is a biased graph that is a single-link extension of $\Omega \setminus \ell$. By comparing 3-element circuits we again get that $F(\Omega') \neq L(\Omega)$, a contradiction. If $\vec{\ell}$ has weight 3, then let X be any subgraph of Ω consisting of an unbalanced 2-cycle along with a pendant link. The columns of A corresponding to X along with the column $\vec{\ell}$ form a circuit of $M(A)$, so again $M(A) \neq L(\Omega)$, a contradiction.

Part 2 Suppose by way of contradiction that \mathbb{F} -matrix A represents $F(\Omega)$ where $A \setminus \vec{\ell} = A_L(G \setminus \ell, \psi)$. Again $\vec{\ell} \neq 0$. Let $\hat{v}_0, \hat{v}_1, \hat{v}_2, \hat{v}_3$ be the rows of A in which \hat{v}_0 corresponds to the gain function ψ . Since $F(\Omega)$ has rank 3 and since $A \setminus \vec{\ell} = A_L(G \setminus \ell, \psi)$ has four rows but only rank 3, it must be that $\vec{\ell}$ keeps the rank of A at 3. This requires that rows $\hat{v}_1, \hat{v}_2, \hat{v}_3$ of $\vec{\ell}$ add to zero. First suppose that $\vec{\ell}$ is zero in rows $\hat{v}_1, \hat{v}_2, \hat{v}_3$. In this case $M(A) = L_0(\Omega \setminus \ell) \neq F(\Omega)$, a contradiction. Second, we may suppose that the entries of $\vec{\ell}$ in rows $\hat{v}_1, \hat{v}_2, \hat{v}_3$ are $a, -a, 0$. In this case $A = A_L(G', \psi')$ where (G', ψ') is obtained from (G, ψ) by the addition of a $v_1 v_2$ -link. There is a 2-cycle on vertices v_1 and v_3 or v_2 and v_3 that is incident to ℓ and this 2-cycle along with ℓ forms a circuit in $F(\Omega)$ but not in $M(A) = M(A_L(G', \psi'))$, a contradiction. Third, assume that the entries of $\vec{\ell}$ in rows $\hat{v}_1, \hat{v}_2, \hat{v}_3$ are $a, b, -(a+b)$ which are all nonzero. In this case, let C be the edge set of a 2-cycle along with a pendant edge. In matroid $M(A)$ there is a circuit in $C \cup \ell$ that must use the pendant edge; however, C may be chosen so that $C \cup \ell$ contains a circuit of $F(\Omega)$ that does not use the pendant edge, a contradiction. \square

Proof of Theorem 5.2. The uniqueness part of our theorem follows from Theorem 5.1. We need only show projective equivalence to an appropriate canonical representation. Let A be an \mathbb{F} -representation of matroid $M \in \{F(G, \mathcal{B}), L(G, \mathcal{B})\}$. By Proposition 3.2 and Theorem 3.4, (G, \mathcal{B}) contains a subgraph (G_0, \mathcal{B}_0) which is a subdivision of a base biased graph or one of $T'_{2,3}$, $T'_{2,2}$, and $T'_{2,1}$. For comparisons with submatrices of A say that the vertex set of G_0 is the same as the vertex set of G . Now let A_0 be the submatrix of A consisting of the columns for $E(G_0)$. By Lemmas 4.13, 4.14, 4.15, 4.16, and 4.17 as well as Proposition 2.9 the matrix A_0 is projectively equivalent to $A_F(G_0, \varphi_0)$ for some φ_0 or $A_L(G_0, \psi_0)$ for some ψ_0 ; that is, $A_F(G_0, \varphi_0) = T_0 A_0 S_0$ or $A_L(G_0, \psi_0) = T_0 A_0 S_0$ for some invertible matrix T_0 and diagonal matrix S_0 .

Let J_G be the collection of joints of (G, \mathcal{B}) . Now, because both G_0 and G are vertically 2-connected (aside for isolated vertices), there is a collection of vertically 2-connected (aside for isolated vertices) biased subgraphs $(G_0, \mathcal{B}_0) \subset \cdots \subset (G_n, \mathcal{B}_n) = (G, \mathcal{B}) \setminus J_G$ and $(G_{i+1}, \mathcal{B}_{i+1}) = (G_i, \mathcal{B}_i) \cup P_i$ for some path P_i in G whose endpoints are in G_i minus isolated vertices and whose internal vertices are isolated in G_i . Also, let A_0, \dots, A_n be the submatrices of A (with the same number of rows as A) corresponding, respectively, to $(G_0, \mathcal{B}_0), \dots, (G_n, \mathcal{B}_n)$. Assume inductively for some $i \geq 0$ that either $A_F(G_j, \varphi_j) = T_j A_j S_j$ for each $j \in \{0, \dots, i\}$ or $A_L(G_j, \psi_j) = T_j A_j S_j$ for each $j \in \{0, \dots, i\}$. In either case T_j is invertible and S_j is diagonal. We will show the same projective equivalence for A_{i+1} . We will obtain this conclusion for the case where P_i is a single edge. If P_i has length greater than one, then the conclusion will follow by Proposition 2.9. Consider $T_i A_{i+1} S'_i$ where S'_i is obtained from S_i by adding an elementary column for the new edge P_i . Let e_i be the column of $T_i A_{i+1} S'_i$ corresponding to the edge P_i . Say that the endpoints of P_i are u_i and v_i . In Case 1 say that $T_i A_i S_i = A_F(G_i, \varphi_i)$ and in Case 2 say that $T_i A_i S_i = A_L(G_i, \psi_i)$.

Case 1 First we show that e_i is zero in every row not corresponding to u_i and v_i . By way of contradiction, say that e_i is nonzero in the row of $T_i A_{i+1} S'_i$ corresponding to vertex $x \in G_i$ where $x \notin \{u_i, v_i\}$. Since (G_0, \mathcal{B}_0) does not have a balancing vertex, neither does (G_i, \mathcal{B}_i) . Thus $(G_i, \mathcal{B}_i) - x$ is unbalanced and connected. Thus there is $U_i \subset (G_i - x)$ consisting of a spanning tree for $G_i - x$ (minus its isolated vertices) along with one additional edge whose fundamental cycle is unbalanced. Thus $U_i \cup P_i$ contains a frame-matroid circuit C passing through P_i and avoiding x . Well, the columns of $T_i A_i S_i = A_F(G_i, \varphi_i)$ corresponding to $C - e_i$ are all zero in the row corresponding to x and these columns are linearly independent. Hence the columns corresponding to C in $T_i A_{i+1} S'_i$ are linearly dependent which contradicts the fact that $T_i A_{i+1} S'_i$ is an \mathbb{F} -representation of $F(G_{i+1}, \mathcal{B}_{i+1})$.

Second we show that both rows of e_i corresponding to u_i and v_i are both nonzero. Since e_i is not a matroid loop, at least one row (say u_i without loss of generality) is nonzero and by way of contradiction assume that row v_i is zero. Now take a subgraph Q of $G_i - v_i$ consisting of an unbalanced cycle along with a path connecting this cycle to u_i (possibly of length zero). Now $Q \cup e_i$ is an independent set in $F(G_{i+1}, \mathcal{B}_{i+1})$ but the columns of $T_i A_{i+1} S'_i$ corresponding to $Q \cup e_i$ are linearly dependent, a contradiction.

By the previous two paragraphs, column e_i of $T_i A_{i+1} S'_i$ is nonzero in exactly the rows corresponding to u_i and v_i and so we let $T_{i+1} = T_i$ and S_{i+1} is obtained from S'_i by changing the scaling factor for e_i so as to make one of the rows equal to 1. We now have that $T_{i+1} A_{i+1} S_{i+1} = A_F(G_{i+1}, \varphi_{i+1})$ and so by induction $T_n A_n S_n = A_F(G_n, \varphi_n)$.

Now if $J_G = \emptyset$, then we are done and so we assume that $J_G = \{l_1, \dots, l_m\}$. Recall than $M \in \{F(G, \mathcal{B}), L(G, \mathcal{B})\}$; however, in this case we must have that $M = F(G, \mathcal{B})$. Suppose by way of contradiction that $M = L(G, \mathcal{B}) \neq F(G, \mathcal{B})$. Since $T_n A_n S_n = A_F(G_n, \varphi_n)$ is a representation of $M \setminus J_G = L(G, \mathcal{B}) \setminus J_G$, it must be that G_n is tangled. Theorem 3.3 implies that (G_n, \mathcal{B}_n) contains as a minor a biased graph (H, \mathcal{S}) that is a biased K_4 or $2C_3$ without a balancing vertex. Thus (G, \mathcal{B}) contains $(H, \mathcal{S}) \cup e_1$ as a minor where e_1 is a joint attached to some vertex. Since $T_n A_n S_n = A_F(G_n, \varphi_n)$, the induced matrix representation of (H, \mathcal{S}) is a frame representation. But now A does not induced a representation of $L((H, \mathcal{S}) \cup e_1) \neq F((H, \mathcal{S}) \cup e_1)$ by Lemma 5.3, a contradiction.

Finally, take any $l_i \in J_G$ and say that the endpoint of l_i is a_i . Let A_{l_i} be the submatrix of

A corresponding to $(G_n, \mathcal{B}_n) \cup l_i$ in which the column for l_i is \vec{l}_i and consider $T_n A_{l_i} S'_n$ where S'_n is S_n with an elementary column added for scaling \vec{l}_i . Say that \vec{l}_i is nonzero in some row $b_i \neq a_i$. Take a subgraph C consisting of an unbalanced cycle in $(G_n, \mathcal{B}_n) \setminus b_i$ along with a path connecting this cycle to a_i . The edges of $C \cup l_i$ form a circuit of $F(G, \mathcal{B})$ but not in the matrix A_{l_i} , a contradiction. Thus \vec{l}_i is zero in the row a_i only. Thus A is projectively equivalent to $A_F(G, \varphi)$ for some φ .

Case 2 Let \hat{v} be the row of A corresponding to the \mathbb{F}^+ -gain function. First we show that e_i is zero in every row not corresponding to u_i , v_i , and \hat{v} . Supposing that e_i is nonzero in row $x \notin \{u_i, v_i, \hat{v}\}$ we use the same technique as in Case 1 to get a lift circuit C in $(G_{i+1}, \mathcal{B}_{i+1})$ such that $e_i \in C$ but C avoids vertex x . The columns of $T_i A_{i+1} S'_i$ are therefore linearly independent and so $T_i A_{i+1} S'_i$ does not represent $L(G_{i+1}, \mathcal{B}_{i+1})$, a contradiction.

Second, the rows u_i and v_i of column e_i must add to zero or matrix $T_i A_{i+1} S'_i$ will have rank one more than should. If rows u_i and v_i of e_i are both 0, then row \hat{v} of e_i must be nonzero. Thus $T_i A_{i+1} S'_i$ is a representation of $L_0(G_i, \mathcal{B}_i)$ as well as of $L(G_{i+1}, \mathcal{B}_{i+1})$. However, if we take any unbalanced cycle C of (G_i, \mathcal{B}_i) that avoids u_i , then C along with the joint element forms a circuit of $L_0(G_i, \mathcal{B}_i)$ but C along with P_i is not a circuit of $L(G_{i+1}, \mathcal{B}_{i+1})$, a contradiction. Thus column e_i has entries a and $-a$ for some $a \neq 0$ in rows u_i and v_i .

By the previous two paragraphs, setting $T_{i+1} = T_i$ and $S_{i+1} = S'_i$ with the column for e_i scaled by $1/a$ we obtain $T_{i+1} A_{i+1} S_{i+1} = A_L(G_{i+1}, \psi_{i+1})$ for some ψ_{i+1} and so by induction $T_n A_n S_n = A_L(G_n, \psi_n)$.

Now if $J_G = \emptyset$, then we are done and so we assume that $J_G = \{l_1, \dots, l_m\}$. Recall that $M \in \{F(G, \mathcal{B}), L(G, \mathcal{B})\}$; however, in this case we must have that $M = L(G, \mathcal{B})$. Suppose by way of contradiction that $M = F(G, \mathcal{B}) \neq L(G, \mathcal{B})$. Since $T_n A_n S_n = A_L(G_n, \varphi_n)$ is a representation of $M \setminus J_G = F(G, \mathcal{B}) \setminus J_G$, it must be that G_n is tangled. Theorem 3.3 implies that (G_n, \mathcal{B}_n) contains as a minor a biased graph (H, \mathcal{S}) that is a biased K_4 or $2C_3$ without a balancing vertex. Thus (G, \mathcal{B}) contains $(H, \mathcal{S}) \cup e_1$ as a minor where e_1 is a joint attached to some vertex. Since $T_n A_n S_n = A_F(G_n, \varphi_n)$, the induced matrix representation of (H, \mathcal{S}) is a lift-matrix representation. But now A does not induced a representation of $F((H, \mathcal{S}) \cup e_1) \neq L((H, \mathcal{S}) \cup e_1)$ by Lemma 5.3, a contradiction.

Finally, take any $l_i \in J_G$ and say that the endpoint of l_i is a_i . Let A_{l_i} be the submatrix of A corresponding to $(G_n, \mathcal{B}_n) \cup l_i$ in which the column for l_i is \vec{l}_i and consider $T_n A_{l_i} S'_n$ where S'_n is S_n with an elementary column added for scaling \vec{l}_i . Say that \vec{l}_i is nonzero in some row aside from \hat{v} , say \hat{u} . There is an unbalanced cycle C in $(G_n, \mathcal{B}_n) - u$ and so $C \cup l_i$ is a circuit of $L((G_n, \mathcal{B}_n) \cup l_i)$, however, the columns of $T_n A_{l_i} S'_n$ corresponding to $C \cup l_i$ are linearly independent, a contradiction. Thus A is projectively equivalent to $A_L(G, \varphi)$ for some φ . \square

Theorem 5.4 (Main Result IV). *Let (G, \mathcal{B}) be a vertically 2-connected, almost-balanced biased graph. If A is an \mathbb{F} -representation of $F(G, \mathcal{B})$ or $L(G, \mathcal{B})$, then A is projectively equivalent to some frame matrix and to some lift matrix specific to (G, \mathcal{B}) or to some biased graph obtained from (G, \mathcal{B}) by a sequence of rolling, unrolling, double rolling, or double unrolling operations.*

Recall that $D_{1,0}$ is the biased K_4 with balanced cycles consisting of exactly one balanced triangle. It has a unique balancing vertex and $D_{1,0} \cong \Delta_X T'_2$. The graph $2C_3 \setminus e$ is obtained from $2C_4'''$ by contracting one of the non-doubled links. The cycles of $2C_3 \setminus e$ are in bijective

correspondence with the cycles of $2C_4''$ and so there are exactly three biased graphs $(2C_3 \setminus e, \mathcal{B})$ without a balanced 2-cycle. These are the single-edge contractions of \mathbf{B}_0 , \mathbf{B}_1 , and \mathbf{B}_2 which we will denote by \mathbf{B}'_0 , \mathbf{B}'_1 , and \mathbf{B}'_2 .

Proposition 5.5. *Let (G, \mathcal{B}) be a vertically 2-connected biased graph that contains a contrabalanced theta subgraph, a unique balancing vertex u , and no joints on vertices other than u . Then (G, \mathcal{B}) contains a subdivision of $\mathbf{D}_{1,0}$, \mathbf{B}'_0 , \mathbf{B}'_1 , or \mathbf{B}'_2 .*

Proof. The graph nK_2 consists of two vertices and n parallel links. Since (G, \mathcal{B}) contains a contrabalanced theta subgraph, it contains a subdivision of (nK_2, \emptyset) for some $n \geq 3$. Let K be such a subdivision in (G, \mathcal{B}) with n as large as possible. One of the two degree- n vertices of K must be u (i.e., the unique balancing vertex of (G, \mathcal{B})) and denote the other degree- n vertex of K by v . Of course K is the union of n internally disjoint uv -paths P_1, \dots, P_n .

Now, since v is not a balancing vertex of (G, \mathcal{B}) , there is some path P in G that is internally disjoint from K , has both endpoints in K , and one of these endpoints is an internal vertex of some P_i . For this proof only, call such a path a K -linker. If there is a K -linker whose endpoints are on internal vertices of two distinct paths P_i and P_j in K , then $K \cup P$ contains a subdivision of $\mathbf{D}_{1,0}$, as required. So assume that no such K -linker exists in G . Since (G, \mathcal{B}) does not have that structure described in Proposition 2.10 (which would result in two distinct balancing vertices in (G, \mathcal{B}) , a contradiction) there must be some K -linker P whose endpoints are u and an internal vertex of some P_i such that $P_i \cup P$ is unbalanced. Thus $K \cup P$ contains a subdivision of \mathbf{B}'_0 , \mathbf{B}'_1 , or \mathbf{B}'_2 , as required. \square

Proof of Theorem 5.4. Let J_G be the set of joints in (G, \mathcal{B}) and let v be the balancing vertex of $(G, \mathcal{B}) - J_G$. If $F(G, \mathcal{B}) = L(G, \mathcal{B})$, then J_G is either empty or consists of loops incident to v . In this case we may as well assume that there is only one joint incident to v because more than one such loop would make parallel elements in the matroid. If $F(G, \mathcal{B}) \neq L(G, \mathcal{B})$, then (G, \mathcal{B}) has joints incident to vertices other than v . If A is a representation of $L(G, \mathcal{B})$, then we can just move any such joints to v without affecting the matroid and then just remove all but one such joint and now $F(G, \mathcal{B}) = L(G, \mathcal{B})$. If A is a representation of $F(G, \mathcal{B})$, then replace (G, \mathcal{B}) with its unrolling at vertex v (denote this biased graph by (G, \mathcal{B}) as well). In this case A is still a representation of $F(G, \mathcal{B})$ but now $F(G, \mathcal{B}) = L(G, \mathcal{B})$. So we may assume that (G, \mathcal{B}) has balancing vertex v with $J_G = \emptyset$ or $J_G = \{\ell\}$ with ℓ incident to v and also that A is a representation of $F(G, \mathcal{B}) = L(G, \mathcal{B})$.

If (G, \mathcal{B}) does not contain a contrabalanced theta subgraph, then (G, \mathcal{B}) is realizable over the group of order 2 and so $F(G, \mathcal{B}) = L(G, \mathcal{B})$ is binary. Hence A is the projectively unique representation of its matroid and so is projectively equivalent to a frame-matrix representation of a roll-up of (G, \mathcal{B}) and to a lift matrix representation particular to (G, \mathcal{B}) . So assume now that (G, \mathcal{B}) contains a contrabalanced theta subgraph. In Case 1 say that $(G, \mathcal{B}) - J_G$ contains a balancing vertex $u \neq v$ and in Case 2 say that v is the unique balancing vertex of $(G, \mathcal{B}) - J_G$.

Case 1 The biased graph (G, \mathcal{B}) has the structure described in Proposition 2.10, either $G = G_1 \cup \dots \cup G_m$ or $G = G_1 \cup \dots \cup G_m \cup \ell$ with $G_i \cap G_j = \{u, v\}$. The parameter m must be at least 3 for (G, \mathcal{B}) to contain a contrabalanced theta subgraph and so u and v are the only balancing vertices of $(G, \mathcal{B}) - J_G$. If $J_G = \{\ell\}$, then replace (G, \mathcal{B}) with unrolling at vertex $u \neq v$. Note that this is a double unrolling of the original biased graph. Now

let (H_0, \mathcal{B}_0) be a subgraph of (G, \mathcal{B}) that consists of a subdivision of (mK_2, \emptyset) obtained by taking a uv -path in each G_i . Since (G, \mathcal{B}) is vertically 2-connected, there is a collection of vertically 2-connected biased subgraphs $(H_0, \mathcal{B}_0) \subset \cdots \subset (H_n, \mathcal{B}_n)$ where $(H_n, \mathcal{B}_n) = (G, \mathcal{B})$ and $(H_{i+1}, \mathcal{B}_{i+1}) = (H_i, \mathcal{B}_i) \cup P_i$ for some path P_i in G . For each H_i , let $G_{j,i} = G_j \cap H_i$. Note that the path P_i must have both endpoints in one $G_{j,i}$. Now for each H_i add in the remaining vertices of G as isolated vertices and let A_i be the submatrix of A corresponding to the edges in H_i .

The matroids of the biased graph (mK_2, \emptyset) has rank 2. Hence any representation of $F(mK_2, \emptyset) = L(mK_2, \emptyset)$ is projectively equivalent to a frame matrix particular to (mK_2, \emptyset) , a roll-up of (mK_2, \emptyset) , or a double roll-up of (mK_2, \emptyset) ; furthermore, the same representation is projectively equivalent to a lift matrix particular to (mK_2, \emptyset) or a roll-up of (mK_2, \emptyset) . Now Proposition 2.9 implies that A_0 is projectively equivalent to a frame matrix particular to a subdivision of (mK_2, \emptyset) , a subdivision of a roll-up of (mK_2, \emptyset) , or a subdivision of double roll-up of (mK_2, \emptyset) . If a roll-up occurs, then one can use row operations to show that this matrix is then projectively equivalent to a roll-up of a subdivision of (mK_2, \emptyset) or a double roll-up of a subdivision of (mK_2, \emptyset) . In other words, A_0 is projectively equivalent a frame matrix particular to (H_0, \mathcal{B}_0) , a roll-up of (H_0, \mathcal{B}_0) , or a double roll-up of (H_0, \mathcal{B}_0) . Similarly A_0 is also projectively equivalent to a lift matrix particular to (H_0, \mathcal{B}_0) or a roll up of (H_0, \mathcal{B}_0) .

Now, inductively assume that $T_0 A_0 S_0, \dots, T_i A_i S_i$ are all canonical matrices (frame or lift) particular to, respectively, $(H_0, \mathcal{B}_0), \dots, (H_i, \mathcal{B}_i)$, some roll-ups of them, or some double roll-ups of them. Consider $(H_{i+1}, \mathcal{B}_{i+1}) = (H_i, \mathcal{B}_i) \cup P_i$; say that the endpoints of P_i are in $G_{j,i}$, call them x_i and y_i . Assume for the moment that P_i has length one, i.e., P_i consists of a single edge, call it e . Since (H_i, \mathcal{B}_i) is vertically 2-connected, one can find a matroid circuit (either a balanced cycle or a contrabalanced theta) that contains P_i and avoids any given $x \notin \{u, v\}$. As such that $T_i A_{i+1}$ has a zero entry in the column for e and row for x . Similarly if $y \notin \{u, v\}$, then any circuit containing P_i forces $T_i A_{i+1}$ to have a nonzero entry in the column for e and the row for y . Note that if $z \in \{u, v\}$ is an endpoint for P_i and the column for e in row z is zero, then it must be that e is in an rolled-up unbalancing class of edges. Hence for an appropriate column-scaling matrix S_{i+1} , we have that $T_i A_{i+1} S_{i+1}$ is a frame matrix particular to $(H_{i+1}, \mathcal{B}_{i+1})$, a roll-up of $(H_{i+1}, \mathcal{B}_{i+1})$, or a double roll-up of $(H_{i+1}, \mathcal{B}_{i+1})$. Now if P_i has length greater than one, then the same conclusion holds by using row operations on the rows corresponding to the vertices of as in the previous paragraph.

Case 2 Since (G, \mathcal{B}) is vertically 2-connected, Proposition 5.5 implies that (G, \mathcal{B}) contains a subgraph (G_0, \mathcal{B}_0) that is a subdivision of $D_{1,0}$, B'_0 , B'_1 , or B'_2 . Add in any of the vertices of G that are missing in G_0 and now because both G_0 and G are vertically 2-connected (aside for isolated vertices), there is a collection of vertically 2-connected (aside for isolated vertices) biased subgraphs $(G_0, \mathcal{B}_0) \subset \cdots \subset (G_n, \mathcal{B}_n)$ where $(G_n, \mathcal{B}_n) = (G, \mathcal{B}) - J_G$ and $(G_{i+1}, \mathcal{B}_{i+1}) = (G_i, \mathcal{B}_i) \cup P_i$ for some path P_i in G whose endpoints are in G_i minus isolated vertices and whose internal vertices are isolated in G_i . Also, let A_0, \dots, A_n be the submatrices of A (with the same number of rows as A) corresponding, respectively, to $(G_0, \mathcal{B}_0), \dots, (G_n, \mathcal{B}_n)$. By Lemma 4.18 and our facts on ΔY -exchanges (note that $\Delta_X D_{1,0} \cong B'_0$), A_0 is projectively equivalent to a frame matrix and projectively equivalent to a lift matrix. Now the result follows via a scheme of appending paths using an induction argument similar to that in the

proof of Theorem 5.2 and in Case 1. □

References

- [1] Tom Brylawski, *Modular constructions for combinatorial geometries*, Trans. Amer. Math. Soc. **203** (1975), 1–44.
- [2] R. Chen and I. Pivotto, *Biased graphs with no two vertex-disjoint unbalanced cycles*, ArXiv e-prints (2014).
- [3] Daryl Funk, *On excluded minors and biased graph representations of frame matroids*, Thesis, Simon Fraser University, Spring 2015.
- [4] Jim Geelen, *Some open problems on excluding a uniform matroid*, Adv. in Appl. Math. **41** (2008), no. 4, 628–637. MR 2459453 (2009k:05050)
- [5] Jim Geelen, Bert Gerards, and Geoff Whittle, *Quasi-graphic matroids*, arXiv:1512.03005, 14pp.
- [6] ———, *The highly connected matroids in minor-closed classes*, Ann. Comb. **19** (2015), no. 1, 107–123.
- [7] J. Kahn and J. P. S. Kung, *Varieties of combinatorial geometries*, Trans. Amer. Math. Soc. **271** (1982), no. 2, 485–499. MR 654846 (84j:05043)
- [8] James Oxley, *Matroid theory*, second ed., Oxford Graduate Texts in Mathematics, vol. 21, Oxford University Press, Oxford, 2011. MR 2849819 (2012k:05002)
- [9] P. D. Seymour, *Decomposition of regular matroids*, J. Combin. Theory Ser. B **28** (1980), no. 3, 305–359. MR 579077 (82j:05046)
- [10] Daniel Slilaty, *Projective-planar signed graphs and tangled signed graphs*, J. Combin. Theory Ser. B **97** (2007), no. 5, 693–717. MR 2344133 (2008j:05161)
- [11] Geoff Whittle, *Stabilizers of classes of representable matroids*, J. Combin. Theory Ser. B **77** (1999), no. 1, 39–72. MR 1710531 (2000f:05027)
- [12] Thomas Zaslavsky, *Vertices of localized imbalance in a biased graph*, Proc. Amer. Math. Soc. **101** (1987), no. 1, 199–204. MR 897095 (88f:05103)
- [13] ———, *Biased graphs. I. Bias, balance, and gains*, J. Combin. Theory Ser. B **47** (1989), no. 1, 32–52. MR 1007712 (90k:05138)
- [14] ———, *Biased graphs. II. The three matroids*, J. Combin. Theory Ser. B **51** (1991), no. 1, 46–72. MR 1088626 (91m:05056)
- [15] Thomas Zaslavsky, *Frame matroids and biased graphs*, European J. Combin. **15** (1994), no. 3, 303–307. MR 1273951 (95a:05021)

- [16] ———, *Biased graphs. IV. Geometrical realizations*, J. Combin. Theory Ser. B **89** (2003), no. 2, 231–297. MR 2017726 (2005b:05057)
- [17] ———, *Biased graphs. VII. Contrabalance and antivoltages*, J. Combin. Theory Ser. B **97** (2007), no. 6, 1019–1040. MR 2354716 (2008h:05025)